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$\mathcal{D}$-compatible semigroup varieties ☆

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Abstract

The $\mathcal{D}$-compatible semigroup varieties will be characterized and described. It is found that a semigroup variety is $\mathcal{D}$-compatible if and only if it is $\mathcal{J}$-compatible. It is shown that a periodic semigroup variety contains at most six maximal $\mathcal{D}$-compatible subvarieties and every $\mathcal{D}$-compatible subvariety is contained in one of these maximal ones. The semigroup varieties which are minimal for not being $\mathcal{D}$-compatible are found: they are all periodic and countably infinite in number. There are six distinct maximal $\mathcal{D}$-compatible pseudovarieties of semigroups. The semigroup varieties and pseudovarieties which are compatible for each of the Green relations are characterized and described. Analogues for varieties and pseudovarieties of monoids are established. It is shown that if a $\mathcal{D}$-compatible variety of monoids contains a nonabelian group, then it is periodic and consists of completely regular monoids only.

1. Introduction

Let $\mathcal{K}$ be any of the Green relations $\mathcal{H}$, $\mathcal{L}$, $\mathcal{R}$, $\mathcal{D}$ or $\mathcal{J}$. A semigroup S is said to be $\mathcal{K}$-*compatible* if $\mathcal{K}$ is a congruence on S. A semigroup variety is said to be $\mathcal{K}$-*compatible* if each of its members is $\mathcal{K}$-compatible. The same terminology will be used for monoids and for pseudovarieties. Instead of "$\mathcal{H}$-compatible" we also use the more accepted "*cryptic*."

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In the present paper we shall give a complete classification of the $\mathcal{D}$-compatible varieties and pseudovarieties. We shall see (Theorem 6.8) that a variety of semigroups is $\mathcal{J}$-compatible if and only if it is $\mathcal{D}$-compatible. We thereby complete the program begun in [4,5] were $\mathcal{H}$-, $\mathcal{L}$- and $\mathcal{R}$-compatible varieties were classified. The large number of papers of the past half century on structural aspects of semigroup theory is testimony for the key importance of the Green relations introduced in 1951 [2]. More often than not the language of equational logic is not sufficiently rich to deal with structural features and it therefore should come as a surprise that the classifications which we proposed to discover are as transparent as they turn out to be.

We shall use [1,3] as general references. We continue to use the notation and terminology of [4,5]. In order to make this paper self-contained we shall gradually recall the definitions.

Semigroup identities are written in the form $u \approx v$, where u and v are words over a fixed countably infinite alphabet X. For $x \in X, n_x(u)$ denotes the number of occurrences of x in the word u. The identity $u \approx v$ is said to be *regular* if for every $x \in X, n_x(u) \neq 0$ if and only if $n_x(v) \neq 0$. The identity $u \approx v$ is said to be *balanced* if for every $x \in X$, $n_x(u) = n_x(v)$. A semigroup variety satisfies only balanced identities if and only if it contains the variety **Com** of all commutative semigroups and is therefore called an *overcommutative* semigroup variety. Every nonbalanced identity has a consequence of the form $x^n \approx x^{n+m}$ for some $n, m \geqslant 1$. A semigroup satisfying such an identity is said to be periodic of bounded period and index. In particular, a semigroup variety in which a nonbalanced identity is satisfied consists of periodic semigroups and is therefore called a *periodic* semigroup variety.

A semigroup S is said to be *combinatorial* (or *$\mathcal{H}$-trivial*) if Green's $\mathcal{H}$-relation on S is the equality. A semigroup variety is called *aperiodic* if all of its members are combinatorial. An aperiodic variety of semigroups cannot contain **Com** since it cannot contain nontrivial groups, and therefore must be periodic. One readily shows that a semigroup variety is aperiodic if and only if it satisfies an identity of the form $x^n \approx x^{n+1}$ for some $n \geqslant 1$ (see [4, Theorem 4.1]). For a semigroup S, let $\mathbf{HSP}(S)$ be the variety generated by S and $Gr(S)$ the set of all group elements of S. Then $\mathbf{HSP}(S)$ is aperiodic if and only if S is periodic of bounded period and index and $Gr(S)$ consists of idempotents only.

The variety of all semigroups will be denoted $\mathbf{S}$ and the variety of all monoids will be denoted $\mathbf{M}$. For any variety $\mathbf{V}$, $L(\mathbf{V})$ stands for the lattice of subvarieties of $\mathbf{V}$. As we have seen, $L(\mathbf{S})$ can be partitioned into two subsets, namely the set of all overcommutative varieties on the one hand and the set of all periodic varieties on the other. For periodic semigroups the Green relations $\mathcal{D}$ and $\mathcal{J}$ coincide and so it is only for overcommutative varieties that a difference may show up.

The strategy for finding the $\mathcal{H}$-, $\mathcal{L}$- and $\mathcal{R}$-compatible semigroup varieties in [4,5] was based on the establishing of an exhaustive list of so-called *forbidden divisors*. We need not work along these lines in the present paper because the abundance of preparatory work which was done in [4,5] allows us to proceed more directly towards our main goals here. Otherwise the plan of the present paper follows the same pattern as in [4,5], and the main theorems of [4,5] find their parallels for the case of the $\mathcal{D}$-relation.

In Section 2 we look at a few semigroups which will play a relevant role later. A few of them are small semigroups which are not $\mathcal{D}$-compatible, others are $\mathcal{D}$-compatible but do not generate a $\mathcal{D}$-compatible variety. Section 3 classifies the periodic $\mathcal{D}$-compatible semigroup varieties. We are then also in the position to classify the periodic semigroup varieties which are $\mathcal{K}$-compatible for each Green relation $\mathcal{K}$ simultaneously; a corresponding result for overcommutative varieties was already obtained in [4]. Section 4 exhibits the periodic semigroup varieties which are minimal for not being $\mathcal{D}$-compatible. It is in Section 6 that we shall see that there are no overcommutative varieties that are minimal for not being $\mathcal{D}$-compatible. Section 5 deals with pseudovarieties of semigroups and gives the analogues of the main results of Sections 3 and 4. Section 6 gives a characterization of the overcommutative $\mathcal{D}$-compatible varieties. Section 7 summarizes the main results for varieties and pseudovarieties of monoids. In Section 8 we initiate the problem of characterizing and classifying the semigroup varieties consisting of semigroups for which two distinct Green relations coincide, and we establish connections with results obtained earlier.

2. Preliminaries

In this section we shall review some finite semigroups which cannot generate a $\mathcal{D}$-compatible variety. Consider the semigroups B_2 and Q with presentation

$$B_2 = \langle a, b \mid aba = a,\ bab = b,\ b^2 = a^2 = ab^2 = b^2a = a^2b = ba^2 \rangle,$$
$$Q = \langle e, f, c \mid ef = f = f^2,\ fe = e = e^2,\ c^2 = c^3 = ce = cf = fc \rangle.$$

Q^* is the left–right dual of Q. The semigroups B_2 and Q were studied in [2]. It is easy to see that these semigroups are themselves not $\mathcal{D}$-compatible. We shall see in Theorem 4.1 that B_2, Q and Q^* generate the only aperiodic varieties which are minimal for not being $\mathcal{D}$-compatible.

Consider the semigroup P with presentation

$$P = \langle e, a \mid e = e^2,\ ea = a,\ ae = a^2 = a^3 \rangle$$

and let P^* be the left–right dual of P. For p a prime,

$$C_p = \langle g \mid g^{p+1} = g \rangle$$

is the cyclic group of order p generated by g. The semigroups P, P^* and the cyclic groups C_p, p prime, are $\mathcal{D}$-compatible. The following reminds us that the class of all $\mathcal{D}$-compatible semigroups is not closed for taking direct products.

Lemma 2.1. *For any prime p, $P^* \times C_p \times P$ is not $\mathcal{D}$-compatible.*

Proof. For the sake of clarity we consider the copy

$$P^* = \langle f, b \mid f = f^2,\ bf = b,\ fb = b^2 = b^3 \rangle$$

of P^*. Since

$$(f, g, e)(f, g, a) = (f, g^2, a),$$
$$(f, g^{p-1}, e)(f, g^2, a) = (f, g, a),$$

we have that $(f, g, a)\mathcal{L}(f, g^2, a)$ in $P^* \times C_p \times P$. However

$$(b, g^p, e)(f, g, a) = (b, g, a),$$
$$(b, g^p, e)(f, g^2, a) = (b, g^2, a),$$

where the $\mathcal{D}$-classes of (b, g, a) and (b, g^2, a) are singletons. $\square$

The semigroups P, P^* and

$$C = \langle e, a \mid e = e^2,\ ea = ae = a,\ a^2 = a^3 \rangle$$

will play an important role in discriminating between the different types of $\mathcal{D}$-compatible varieties (see Theorems 3.11 and 3.12). Note that P is a homomorphic image of Q and a subsemigroup of B_2. We also remark that, in view of Lemma 2.1, a $\mathcal{D}$-compatible variety which is not aperiodic cannot contain both P and P^*.

The following five element band with presentation

$$B = \langle e, f, c \mid ef = f = f^2,\ fe = e = e^2,\ c^2 = c = ce = cf \rangle$$

was also investigated in [5]. For any prime p we consider the semigroups

$$\begin{aligned}
L(p) &= \langle e, g \mid eg = e = e^2 = g^p e,\ g^{p+1} = g \rangle,\\
L_1(p) &= \langle e, g \mid eg = e = g^p e,\ g^{p+1} = g,\ e^2 = e^3 = ge^2 \rangle,\\
L_2(p) &= \langle e, g \mid e = g^p e,\ g^{p+1} = g,\ e^2 = e^3 = eg = ge^2 \rangle,
\end{aligned}$$

which were also investigated in [5]. The left–right duals of $B, L(p), L_1(p), L_2(p)$ will be denoted by B^*, $R(p)$, $R_1(p)$ and $R_2(p)$, respectively. In Corollary 3.3 we shall see that $P \times B^*$, $P \times R(p)$, p prime, and their duals each generate a variety which is not $\mathcal{D}$-compatible. For the time being, we have

Lemma 2.2. *For p prime,* $\mathbf{HSP}(L_1(p))$ *is not $\mathcal{D}$-compatible.*

Proof. Let S be the semigroup with presentation

$$\langle a, b, g \mid g^{p+1} = g,\ ga = a = ag^p,\ g^p b = b = bg,\ a^2 = b^2 = ba = a^3 = b^3 = ag^i b,\\ i \not\equiv 0 \ (\mathrm{mod}\ p) \rangle.$$

We shall denote the zero $a^2 = b^2$ of S by 0. Then S consists of the $3p+2$ elements $g, g^2, \ldots, g^p$, $a, ag, \ldots, ag^{p-1}, b, gb, \ldots, g^{p-1}b, ab$ and 0. The semigroup S is not $\mathcal{D}$-compatible since $a\mathcal{R}ag$ in S but ab and $0 = agb$ are not $\mathcal{D}$-related. We shall show that $\mathbf{HSP}(S) = \mathbf{HSP}(L_1(p))$. The subsemigroup of S generated by b and g is isomorphic to $L_1(p)$ and so $\mathbf{HSP}(L_1(p)) \subseteq \mathbf{HSP}(S)$. To prove the reverse inclusion we shall show that S satisfies every identity which is satisfied by $L_1(p)$. Such identities are characterized in Lemma 2.1 of [4].

Let $v \approx w$ be any identity which is satisfied by $L_1(p)$, let Y be the set of variables which occur in $v \approx w$, and let $\varphi : Y \to S$ be any evaluation. If $0 \in Y\varphi$, then $v\varphi = w\varphi$ holds true in S because $v \approx w$ is regular by Lemma 2.1 of [4]. We shall henceforth assume that $0 \notin Y\varphi$. The semigroup generated by b and g is isomorphic to $L_1(p)$, the semigroup generated by a and g is isomorphic to $R_1(p)$, and by Lemma 2.1 of [4], $\mathbf{HSP}(L_1(p)) = \mathbf{HSP}(R_1(p))$. Therefore if $Y\varphi$ is contained in either one of these two subsemigroups of S, then $v\varphi = w\varphi$ holds true in S. It remains to investigate the case where $ag^i, g^j b \in Y\varphi$ for some i, j. We have that $v\varphi \neq 0$ only if $v\varphi = ab$, and this is possible only if for some $x, y \in Y$, $v \equiv v_1 x v_2 y v_3$, with $n_x(v) = n_y(v) = 1$, and some integers i_1, i_2, i_3, $v_1\varphi = g^{i_1}$, $x\varphi = ag^i$, $v_2\varphi = g^{i_2}$, $y\varphi = g^j b$, $v_3 = g^{i_3}$, and $i + i_2 + j \equiv$

0 (mod p). By Lemma 2.1 of [4] we then have $w \equiv w_1 x w_2 y w_3$ where $n_x(w) = n_y(w) = 1$, and for some integers j_1, j_2, j_3, $w_1\varphi = g^{j_1}$, $w_2\varphi = g^{j_2}$, $w_3\varphi = g^{j_3}$, with $j_2 \equiv i_2 \pmod p$. It follows that $i + j_2 + j \equiv 0 \pmod p$, so $(xw_2y)\varphi = ag^{i+j_2+j}b = ab$, and thus also $w\varphi = ab = v\varphi$. By symmetry, $w\varphi \neq 0$ if and only if $w\varphi = ab = v\varphi$. $\square$

Every periodic semigroup S can be considered as a semigroup equipped with a unary operation: for $a \in S$, a^0 denotes the idempotent of the cyclic semigroup generated by a. If S is of bounded period and index then S satisfies an identity of the form $x^n \approx x^{n+m}$ for some $m, n \geqslant 1$, and then $a^0 = a^{mn}$ for every $a \in S$.

There are, up to isomorphism, exactly two five element completely 0-simple semigroups which are not completely regular. The semigroup B_2 is one of them, the other we denote by A_2.

We shall have multiple occasions to apply the following lemma. (See Lemma 2.1 of [5].)

Lemma 2.3. *For a periodic semigroup S the following are equivalent*:

(i) *every regular semigroup which divides S is completely regular,*
(ii) *the principal factors of S are null, or completely simple, or completely simple with a zero adjoined,*
(iii) *S does not have a divisor of the form A_2 or B_2,*
(iv) *S satisfies the identities*

$$(xy)^0x\big((xy)^0x\big)^0 \approx (xy)^0x \approx \big((xy)^0x\big)^0(xy)^0x.$$

We remark that if a periodic semigroup is $\mathcal{D}$-compatible then every regular $\mathcal{D}$-class of S must be a subsemigroup of S and thus in fact a completely simple subsemigroup of S. Therefore $\mathcal{D}$-compatible periodic semigroups satisfy the equivalent conditions of Lemma 2.3.

3. Maximal $\mathcal{D}$-compatible periodic semigroup varieties

We shall prove that each periodic semigroup variety has at most six maximal $\mathcal{D}$-compatible subvarieties. To prove this, it suffices to investigate the $\mathcal{D}$-compatible subvarieties of the variety $\mathbf{V}_{n,m}$ which is determined by the identity

$$x^n \approx x^{n+m}. \tag{1}$$

The following subvarieties of $\mathbf{V}_{n,m}$ will be relevant in our considerations. We let $\mathbf{R}_{n,m}$ be the subvariety of $\mathbf{V}_{n,m}$ determined by the additional identities

$$\big(x^{mn+1}y\big)^{mn+1} \approx x^{mn+1}y, \tag{2}$$

$$\big(\big(x^{mn}\big)^{mn}x^{mn}z\big)^{mn}\big(\big(x^{mn}y\big)^{mn+1}z\big)^{mn} \approx \big(\big(x^{mn}y\big)^{mn+1}z\big)^{mn}. \tag{3}$$

We let $\mathbf{RC}_{n,m}$ be the subvariety of $\mathbf{V}_{n,m}$ determined by the additional identity

$$x^{mn+1}y \approx x^{mn}yx^{mn+1}. \tag{4}$$

We let $\mathbf{RI}_{n,m}$ be the subvariety of $\mathbf{V}_{n,m}$ determined by the additional identities

$$\big((xy)^{mn}x\big)^{mn}z \approx (xy)^{mn}z, \tag{5}$$

$$(xy)^{mn}x\big((xy)^{mn}x\big)^{mn} \approx (xy)^{mn}x \approx \big((xy)^{mn}x\big)^{mn}(xy)^{mn}x. \tag{6}$$

The left–right duals of $\mathbf{R}_{n,m}, \mathbf{RC}_{n,m}$ and $\mathbf{RI}_{n,m}$ will be denoted by $\mathbf{L}_{n,m}, \mathbf{LC}_{n,m}$ and $\mathbf{LI}_{n,m}$, respectively. We let $\mathbf{D}_{n,m}$ be the subvariety of $\mathbf{V}_{n,m}$ which is determined by the additional identity (2) and its dual.

The following provides more details concerning the variety $\mathbf{RI}_{n,m}$.

Lemma 3.1.

(i) *Let $S \in \mathbf{V}_{n,m}$. Then $S \in \mathbf{RI}_{n,m}$ if and only if the regular $\mathcal{D}$-classes of S form completely simple semigroups, and, if e and f are $\mathcal{R}$-related idempotents of S, and $a \in S$, then $ea = fa$.*

(ii) *For $n > 1$, $\mathbf{RI}_{n,m}$ is the greatest subvariety of $\mathbf{V}_{n,m}$ which contains P but not Q nor B_2.*

Proof. (i) Assume that $S \in \mathbf{RI}_{n,m}$. It follows immediately from Lemma 2.3 that the regular $\mathcal{D}$-classes of S form completely simple semigroups. Let e and f be $\mathcal{R}$-related idempotents of S, and $a \in S$. Then since S satisfies (5),

$$ea = \big((ef)^{mn}e\big)^{mn}a = (ef)^{mn}a = fa.$$

Assume conversely that $S \in \mathbf{V}_{n,m}$ such that every regular $\mathcal{D}$-class of S forms a completely simple semigroup and such that for any $\mathcal{R}$-related idempotents e and f and $a \in S$, $ea = fa$ holds. From Lemma 2.3 we have that S satisfies (6). For $a, b, c \in S$ we have that $((bc)^{mn}b)^{mn}$ and $(bc)^{mn}$ are $\mathcal{R}$-related idempotents of S, and thus

$$\big((bc)^{mn}b\big)^{mn}a = (bc)^{mn}a.$$

Therefore S satisfies (5). Consequently $S \in \mathbf{RI}_{n,m}$.

(ii) That $\mathbf{RI}_{n,m}$ does not contain Q nor B_2 follows from (i). That $\mathbf{RI}_{n,m}$ contains P also follows from (i). Let $\mathbf{V}$ be any subvariety of $\mathbf{V}_{n,m}$ which contains P but not Q nor B_2. By Lemma 2.3, and Lemma 2.11 of [5], the identities (6) are satisfied in $\mathbf{V}$ and $\mathbf{V}$ consists of semigroups whose regular $\mathcal{D}$-classes are completely simple semigroups. As in the proof for Lemma 4.4 of [5] we can show that there exists a regular identity $u \approx v$ which is satisfied in $\mathbf{V}$, such that u and v end in the same variable x, $n_x(u) = 1 = n_x(v)$, and such that u and v end in yx and zx respectively for distinct variables y and z. If e and f are $\mathcal{R}$-related idempotents of $S \in \mathbf{V}$ and $a \in S$, consider a substitution $\varphi : X \to S$, where $x\varphi = a$, $y\varphi = e$, $z\varphi = f$, and $q\varphi = e$ for all other variables. Then $ea = u\varphi = v\varphi = f\varphi$ since S satisfies $u \approx v$. Therefore by (i), $S \in \mathbf{RI}_{n,m}$. Hence $\mathbf{V} \subseteq \mathbf{RI}_{n,m}$, as required. □

As in [2] we let $\mathbf{RAp}_n$ be the semigroup variety determined by the identities

$$x^n \approx x^{n+1}, \tag{7}$$

$$(xy)^n z \approx (xy)^n xz. \tag{8}$$

$\mathbf{LAp}_n$ will denote the left–right dual of $\mathbf{RAp}_n$. From Theorem 4.1(ii) of [5] and Lemma 3.1 we thus have

Corollary 3.2. $\mathbf{RAp}_n = \mathbf{V}_{n,1} \cap \mathbf{RI}_{n,m}$, $\mathbf{LAp}_n = \mathbf{V}_{n,1} \cap \mathbf{LI}_{n,m}$.

Another consequence of Lemma 3.1 is the following useful

Corollary 3.3.

(i) $Q \in \mathbf{HSP}(P \times B^*)$,
(ii) $Q \in \mathbf{HSP}(P \times R(p))$.

Proof. We have that $B^* \notin \mathbf{RI}_{n,m}$ by Lemma 3.1(i) since ce and cf are distinct $\mathcal{R}$-related idempotents of B^* and $(ce)e = ce \neq cf = (cf)e$. Further, $B_2 \notin \mathbf{HSP}(P \times B^*)$ since $P \times B^*$ satisfies $y(xy)^2 \approx (y(xy)^2)^2$ while B_2 does not. Using Lemma 3.1(ii), if $Q \notin \mathbf{HSP}(P \times B^*)$, then since $P \in \mathbf{HSP}(P \times B^*)$ and $B_2 \notin \mathbf{HSP}(P \times B^*)$, we would have that $\mathbf{HSP}(P \times B^*) \subseteq \mathbf{RI}_{n,m}$ for any $n > 1$ by Lemma 3.1(ii), and this is impossible since $B^* \notin \mathbf{RI}_{n,m}$ for any n.

We have that $R(p) \notin \mathbf{RI}_{n,m}$ by Lemma 3.1(ii) since in $R(p)$, e and eg are distinct $\mathcal{R}$-related idempotents, and g^p an idempotent where $eg^p = e \neq eg = egg^p$. Also $B_2 \notin \mathbf{HSP}(P \times R(p))$ since $P \times R(p)$ satisfies $y(xy)^2 \approx (y(xy)^2)^{p+1}$, while B_2 does not. Again as above, $Q \notin \mathbf{HSP}(P \times R(p))$ would lead to a contradiction. □

The varieties $\mathbf{R}_{n,m}$, $\mathbf{RAp}_n$ and $\mathbf{RC}_{n,m}$ are for $n > 1$, the maximal $\mathcal{R}$-compatible subvarieties of $\mathbf{V}_{n,m}$, while $\mathbf{R}_{1,m}$ is the greatest $\mathcal{R}$-compatible subvariety of $\mathbf{V}_{1,m}$ [5, Theorem 4.6]. The structural properties of the members of these varieties were made clear in [5]. The following is concerned with $\mathbf{D}_{n,m}$.

Theorem 3.4. *$\mathbf{D}_{n,m}$ consists of the semigroups $S \in \mathbf{V}_{n,m}$ such that $Gr(S)$ is an ideal of S.*

Proof. Let $S \in \mathbf{V}_{n,m}$. Then S satisfies (2) if and only if $Gr(S)$ is a right ideal of S (see also the proof of Theorem 4.1(ii) of [5]), since $a \in Gr(S)$ if and only if $a = a^{mn+1}$. □

In order to be able to fabricate $\mathcal{D}$-compatible varieties from the ones listed so far in this section, we need some auxiliary results.

Lemma 3.5. *Let S be any semigroup. Then S is $\mathcal{D}$-compatible if and only if the following two conditions hold*:

(i) *if $a\mathcal{R}b$ in S then $ac\mathcal{D}bc$ in S for all $c \in S$,*
(ii) *if $a\mathcal{L}b$ in S then $ca\mathcal{D}cb$ in S for all $c \in S$.*

Proof. Obviously if S is $\mathcal{D}$-compatible, then (i) and (ii) hold. Conversely, assume that S satisfies (i) and (ii) and let $a\mathcal{D}d$ in S, and $c \in S$. There exists $b \in S$ such that $a\mathcal{R}b\mathcal{L}d$. Then $ac\mathcal{D}dc$ since (i) holds and since $\mathcal{L}$ is a right congruence. Hence $\mathcal{D}$ is a right congruence relation. Dually, $\mathcal{D}$ is also a left congruence relation, and thus a congruence relation. □

Lemma 3.6. *Let $\mathbf{V}$ be an $\mathcal{R}$-compatible subvariety of $\mathbf{V}_{n,m}$. Then $\mathbf{V} \cap \mathbf{LI}_{n,m}$ is $\mathcal{D}$-compatible.*

Proof. Let $S \in \mathbf{V} \cap \mathbf{LI}_{n,m}$ and assume that $a\mathcal{L}b$ in S, with $a \neq b$. Then there exist $s, t \in S$ such that $sa = b$ and $tb = a$, thus $(ts)^{mn}a = a = t(st)^{mn}b$, $(st)^{mn}b = b = s(ts)^{mn}a$. Since S satisfies

(1) and the dual of (6) we have from Lemma 2.3 that $s(ts)^{mn}\mathcal{H}(s(ts)^{mn})^{mn}\mathcal{L}(ts)^{mn}$ in S, where $(s(ts)^{mn})^{mn}$ and $(ts)^{mn}$ are idempotents. Using the dual of Lemma 3.1, we have that

$$c\big(s(ts)^{mn}\big)^{mn}=c(ts)^{mn}$$

for all $c\in S$, and therefore also

$$cs(ts)^{mn}\mathcal{R}c\big(s(ts)^{mn}\big)^{mn}=c(ts)^{mn}$$

for all $c\in S$. Since S is $\mathcal{R}$-compatible we thus obtain

$$cb=c\big(s(ts)^{mn}\big)a\mathcal{R}c(ts)^{mn}a=ca,$$

for all $c\in S$. Therefore S is $\mathcal{D}$-compatible by Lemma 3.5. □

Theorem 3.7. *The following are $\mathcal{D}$-compatible subvarieties of $\mathbf{V}_{n,m}$: $\mathbf{LAp}_n\cap\mathbf{RAp}_n$, $\mathbf{D}_{n,m}$, $\mathbf{R}_{n,m}\cap\mathbf{LI}_{n,m}$, $\mathbf{L}_{n,m}\cap\mathbf{RI}_{n,m}$, $\mathbf{RC}_{n,m}\cap\mathbf{LI}_{n,m}$, $\mathbf{LC}_{n,m}\cap\mathbf{RI}_{n,m}$.*

Proof. From Theorem 4.1 of [5] we know that $\mathbf{R}_{n,m}$, $\mathbf{RAp}_n$ and $\mathbf{RC}_{n,m}$ are $\mathcal{R}$-compatible subvarieties of $\mathbf{V}_{n,m}$. Therefore, from Lemma 3.5 it follows that $\mathbf{R}_{n,m}\cap\mathbf{LI}_{n,m}$, $\mathbf{RAp}_n\cap\mathbf{LI}_{n,m}$ and $\mathbf{RC}_{n,m}\cap\mathbf{LI}_{n,m}$ are $\mathcal{D}$-compatible subvarieties of $\mathbf{V}_{n,m}$. Dually, $\mathbf{L}_{n,m}\cap\mathbf{RI}_{n,m}$ and $\mathbf{LC}_{n,m}\cap\mathbf{RI}_{n,m}$ are $\mathcal{D}$-compatible, while $\mathbf{RAp}_n\cap\mathbf{LI}_{n,m}=\mathbf{RAp}_n\cap\mathbf{V}_{n,1}\cap\mathbf{LI}_{n,m}=\mathbf{RAp}_n\cap\mathbf{LAp}_n$ by Corollary 3.2. It remains to prove that $\mathbf{D}_{n,m}$ is $\mathcal{D}$-compatible.

Let $S\in\mathbf{D}_{n,m}$. Therefore $Gr(S)$ is an ideal of S by Theorem 3.4. Assume that $a\in Gr(S)$ and $c\in S$. For any $k\geqslant 1$,

$$ac^k=\big(ac^k\big)^{mn+1}\mathcal{L}cac^k=\big(cac^k\big)^{mn+1}\mathcal{R}cac^{k+1}\mathcal{L}\big(ac^{k+1}\big)^{mn+1}=ac^{k+1},$$

hence $ac\mathcal{D}ac^{mn}$ in S, where $c^{mn}\in Gr(S)$. Let a and b be distinct $\mathcal{D}$-related elements of S. Then, since $Gr(S)$ is an ideal of S it easily follows that $a,b\in Gr(S)$ and $a\mathcal{D}b$ in $Gr(S)$. Thus, for all $c\in S$, $ac^{mn}\mathcal{D}bc^{mn}$ in $Gr(S)$ and in S, since $\mathcal{D}$ is a congruence relation for completely regular semigroups. From the above it thus follows that $ac\mathcal{D}bc$ in S. We proved that $\mathcal{D}$ is a right congruence relation. By symmetry, $\mathbf{D}_{n,m}$ is $\mathcal{D}$-compatible. □

We set out to prove that every $\mathcal{D}$-compatible periodic semigroup variety is contained in one of the varieties mentioned in the statement of Theorem 3.7 for some positive integers m,n.

Lemma 3.8. *Let $\mathbf{V}$ be a $\mathcal{D}$-compatible subvariety of $\mathbf{V}_{n,m}$ which contains P and P^*. Then $\mathbf{V}\subseteq\mathbf{LAp}_n\cap\mathbf{RAp}_n$.*

Proof. By Lemma 2.1, $\mathbf{V}$ cannot contain a cyclic group C_p for any prime number p. It follows that $\mathbf{V}\subseteq\mathbf{V}_{n,1}$. The variety $\mathbf{V}$ is $\mathcal{D}$-compatible and therefore $\mathbf{V}$ cannot contain Q, Q^* nor B_2. By Lemma 3.1(ii) and its dual, $\mathbf{V}\subseteq\mathbf{LI}_{n,m}\cap\mathbf{RI}_{n,m}$. Therefore $\mathbf{V}\subseteq\mathbf{LI}_{n,m}\cap\mathbf{V}_{n,1}\cap\mathbf{RI}_{n,m}=\mathbf{LAp}_n\cap\mathbf{RAp}_n$ in view of Corollary 3.2. □

Lemma 3.9. *Let $\mathbf{V}$ be a $\mathcal{D}$-compatible subvariety of $\mathbf{V}_{n,m}$ which contains neither P nor P^*. If $C\notin\mathbf{V}$, then $\mathbf{V}\subseteq\mathbf{D}_{n,m}$. If $C\in\mathbf{V}$, then $\mathbf{V}\subseteq(\mathbf{LC}_{n,m}\cap\mathbf{RI}_{n,m})\cap(\mathbf{RC}_{n,m}\cap\mathbf{LI}_{n,m})=\mathbf{LC}_{n,m}\cap\mathbf{RC}_{n,m}$.*

Proof. If $\mathbf{V}$ does not contain C, then $\mathbf{V} \subseteq \mathbf{D}_{n,m}$ by Lemma 2.5 of [4] and Theorem 3.4. Let us now assume that $C \in \mathbf{V}$. Then $\mathbf{V}$ does not contain LZ_2 nor RZ_2, since P and P^* divide $C \times RZ_2$ and $C \times LZ_2$, respectively, by Lemma 2.13(i) of [5] and its dual. Therefore $\mathbf{V}$ cannot contain the semigroups B, nor $L(p)$, p prime, since these semigroups contain LZ_2 as a subsemigroup. Also $\mathbf{V}$ does not contain Q nor B_2 since $\mathbf{V}$ is $\mathcal{D}$-compatible. Further, $\mathbf{V}$ cannot contain $L_1(p)$, p prime, by Lemma 2.2 and also not $L_2(p)$, p prime, since P divides such semigroups $L_2(p)$. By Theorem 3.6 of [5], $\mathbf{V}$ is $\mathcal{R}$-compatible, and by Theorem 4.5(iii) of [5], $\mathbf{V} \subseteq \mathbf{RC}_{n,m}$. By symmetry we also have $\mathbf{V} \subseteq \mathbf{LC}_{n,m}$.

If $S \in \mathbf{LC}_{n,m} \cap \mathbf{RC}_{n,m}$ then by Corollary 4.2 of [5] and its dual, $Gr(S)$ is a semilattice of abelian groups. By Lemma 3.1(i) and its dual, $S \in \mathbf{LI}_{n,m} \cap \mathbf{RI}_{n,m}$. We proved that

$$(\mathbf{LC}_{n,m} \cap \mathbf{RI}_{n,m}) \cap (\mathbf{RC}_{n,m} \cap \mathbf{LI}_{n,m}) = \mathbf{LC}_{n,m} \cap \mathbf{RC}_{n,m}. \qquad \square$$

Lemma 3.10. *Let* $\mathbf{V}$ *be a* $\mathcal{D}$*-compatible semigroup variety which contains* P^* *but not* P*. Then* $\mathbf{V}$ *is* $\mathcal{R}$*-compatible. If moreover* $\mathbf{V} \subseteq \mathbf{V}_{n,m}$, $n > 1$*, then* $\mathbf{V} \subseteq \mathbf{LI}_{n,m}$.

Proof. Since $\mathbf{V}$ is $\mathcal{D}$-compatible, $\mathbf{V}$ contains neither Q nor B_2. Since $\mathbf{V}$ contains P^*, $\mathbf{V}$ cannot contain B nor $L(p)$ for any prime p by the dual of Corollary 3.3. Also, since $\mathbf{HSP}(L_1(p))$ is not $\mathcal{D}$-compatible for any prime p by Lemma 2.2, $\mathbf{V}$ cannot contain $L_1(p)$ for any prime p. Since P is a homomorphic image of $L_2(p)$ for any prime p, and $P \notin \mathbf{V}$, we have that $L_2(p) \notin \mathbf{V}$ for any prime p. By Theorem 6.3 of [5], $\mathbf{V}$ is $\mathcal{R}$-compatible.

If $\mathbf{V} \subseteq \mathbf{V}_{n,m}$, $n > 1$, then $\mathbf{V} \subseteq \mathbf{LI}_{n,m}$ by the dual of Lemma 3.1(ii). $\square$

We summarize our findings in the following two theorems.

Theorem 3.11. *Let* $\mathbf{V} \subseteq \mathbf{V}_{n,1}$, $n > 1$*, be a* $\mathcal{D}$*-compatible variety. Then exactly one of the following occurs*:

(i) $\mathbf{V}$ *contains* C*, or* $\mathbf{V}$ *contains* P *and* P^**, and then* $\mathbf{V} \subseteq \mathbf{LAp}_n \cap \mathbf{RAp}_n$.
(ii) $\mathbf{V}$ *contains* P^* *but neither* P *nor* C*, and then* $\mathbf{V} \subseteq \mathbf{R}_{n,1} \cap \mathbf{LAp}_n$.
(iii) $\mathbf{V}$ *contains* P *but neither* P^* *nor* C*, and then* $\mathbf{V} \subseteq \mathbf{L}_{n,1} \cap \mathbf{RAp}_n$.
(iv) $\mathbf{V}$ *contains neither* P, P^* *nor* C*, and then* $\mathbf{V} \subseteq \mathbf{D}_{n,1}$.

For $n > 1$, $\mathbf{V}_{n,1}$ *has four distinct maximal* $\mathcal{D}$*-compatible subvarieties, namely* $\mathbf{LAp}_n \cap \mathbf{RAp}_n$, $\mathbf{R}_{n,1} \cap \mathbf{LAp}_n$, $\mathbf{L}_{n,1} \cap \mathbf{RAp}_n$ *and* $\mathbf{D}_{n,1}$.

Proof. If $\mathbf{V}$ contains P and P^*, then $\mathbf{V} \subseteq \mathbf{LAp}_n \cap \mathbf{RAp}_n$ by Lemma 3.8. If $\mathbf{V}$ contains C but neither P nor P^*, then $\mathbf{V} \subseteq \mathbf{LC}_{n,1} \cap \mathbf{RC}_{n,1}$, by Lemma 3.9, whereas $\mathbf{LC}_{n,1} \cap \mathbf{RC}_{n,1} \subseteq \mathbf{LAp}_n \cap \mathbf{RAp}_n$ by Theorem 4.1(ii) and Corollary 4.2 of [5] and their duals. If $\mathbf{V}$ contains C and P^*, but not P, then by Theorem 4.5(iii) of [5], Corollary 3.2 and Lemma 3.10 we have that $\mathbf{V} \subseteq \mathbf{RC}_{n,1} \cap \mathbf{LAp}_n$. By Theorem 4.1(ii) and Corollary 4.2 of [5] we have that $\mathbf{RC}_{n,1} \subseteq \mathbf{RAp}_n$ so that again $\mathbf{V} \subseteq \mathbf{LAp}_n \cap \mathbf{RAp}_n$ in this case. By symmetry, if $\mathbf{V}$ contains C and P but not P^*, then again $\mathbf{V} \subseteq \mathbf{LAp}_n \cap \mathbf{RAp}_n$. We proved that if $\mathbf{V}$ satisfies the conditions stipulated in (i), then $\mathbf{V} \subseteq \mathbf{LAp}_n \cap \mathbf{RAp}_n$.

Assume next that $\mathbf{V}$ contains P^* but neither P nor C. By Theorem 4.5(i) of [5], Corollary 3.2 and Lemma 3.10 we have that $\mathbf{V} \subseteq \mathbf{R}_{n,1} \cap \mathbf{LAp}_n$. Dually, if $\mathbf{V}$ contains P but not P^* nor C, then $\mathbf{V} \subseteq \mathbf{L}_{n,1} \cap \mathbf{RAp}_n$.

Finally, assume that $\mathbf{V}$ contains neither P, P^* nor C. Then $\mathbf{V} \subseteq \mathbf{D}_{n,1}$ by Lemma 3.9.

From Theorem 3.7 we know that the four varieties $\mathbf{LAp} \cap \mathbf{RAp}_n$, $\mathbf{R}_{n,1} \cap \mathbf{LAp}_n$, $\mathbf{L}_{n,1} \cap \mathbf{RAp}_n$ and $\mathbf{D}_{n,1}$ are $\mathcal{D}$-compatible. In view of the above, to show that they are the four distinct maximal $\mathcal{D}$-compatible subvarieties of $\mathbf{V}_{n,1}$, it suffices to show that they are pairwise incomparable. Indeed, applying Theorem 4.1 of [5] and Theorem 3.4 we have the following. $\mathbf{LAp}_n \cap \mathbf{RAp}_n$ contains $P \times P^*$ whereas the other three varieties do not. $P^* \times RZ_2^1$ is in $\mathbf{R}_{n,1} \cap \mathbf{LAp}_n$ but not in the other three, and $P \times LZ_2^1$ is in $\mathbf{L}_{n,1} \cap \mathbf{RAp}_n$ but not in the other three. Finally, $B \times B^*$ is in $\mathbf{D}_{n,1}$ but not in the other three. □

Theorem 3.12. *Let $\mathbf{V} \subseteq \mathbf{V}_{n,m}$, $n > 1$, $m > 1$, be a $\mathcal{D}$-compatible variety which is not aperiodic. Then exactly one of the following occurs*:

- (i) $\mathbf{V}$ *contains C but not P, and then $\mathbf{V} \subseteq \mathbf{RC}_{n,m} \cap \mathbf{LI}_{n,m}$.*
- (ii) $\mathbf{V}$ *contains C but not P^*, and then $\mathbf{V} \subseteq \mathbf{LC}_{n,m} \cap \mathbf{RI}_{n,m}$.*
- (iii) $\mathbf{V}$ *contains P^* but neither C nor P, and then $\mathbf{V} \subseteq \mathbf{R}_{n,m} \cap \mathbf{LI}_{n,m}$.*
- (iv) $\mathbf{V}$ *contains P but neither C nor P^*, and then $\mathbf{V} \subseteq \mathbf{L}_{n,m} \cap \mathbf{RI}_{n,m}$.*
- (v) $\mathbf{V}$ *contains neither P, P^* nor C, and then $\mathbf{V} \subseteq \mathbf{D}_{n,m}$.*

For $n, m > 1$, $\mathbf{V}_{n,m}$ has six distinct maximal $\mathcal{D}$-compatible subvarieties, namely $\mathbf{LAp}_n \cap \mathbf{RAp}_n$, $\mathbf{LC}_{n,m} \cap \mathbf{RI}_{n,m}$, $\mathbf{RC}_{n,m} \cap \mathbf{LI}_{n,m}$, $\mathbf{L}_{n,m} \cap \mathbf{RI}_{n,m}$, $\mathbf{R}_{n,m} \cap \mathbf{LI}_{n,m}$, $\mathbf{D}_{n,m}$.

Proof. Since $\mathbf{V}$ is not aperiodic we have that $C_p \in \mathbf{V}$ for some prime divisor p of m. Therefore $\mathbf{V}$ cannot contain both P and P^* by Lemma 2.1, and from this it is easy to see that exactly one of the above five cases arises.

First assume that $\mathbf{V}$ contains C but not P. If $\mathbf{V}$ contains P^*, then $\mathbf{V} \subseteq \mathbf{RC}_{n,m} \cap \mathbf{LI}_{n,m}$ by Theorem 4.5(iii) of [5] and Lemma 3.10. If $\mathbf{V}$ does not contain P^*, then again $\mathbf{V} \subseteq \mathbf{RC}_{n,m} \cap \mathbf{LI}_{n,m}$ by Lemma 3.9. Dually, if $\mathbf{V}$ contains C but not P^*, then $\mathbf{V} \subseteq \mathbf{LC}_{n,m} \cap \mathbf{RI}_{n,m}$.

If $\mathbf{V}$ contains P^* but neither C nor P, then by Theorem 4.5(i) of [5] and Lemma 3.10, $\mathbf{V} \subseteq \mathbf{R}_{n,m} \cap \mathbf{LI}_{n,m}$. Dually, if $\mathbf{V}$ contains P but neither C nor P^*, then $\mathbf{V} \subseteq \mathbf{L}_{n,m} \cap \mathbf{RI}_{n,m}$. Finally, if $\mathbf{V}$ contains neither P, P^* nor C, then $\mathbf{V} \subseteq \mathbf{D}_{n,m}$ by Lemma 3.9.

From Theorem 3.7 we know that the varieties $\mathbf{LAp}_n \cap \mathbf{RAp}_n$, $\mathbf{LC}_{n,m} \cap \mathbf{RI}_{n,m}$, $\mathbf{RC}_{n,m} \cap \mathbf{LI}_{n,m}$, $\mathbf{L}_{n,m} \cap \mathbf{RI}_{n,m}$, $\mathbf{R}_{n,m} \cap \mathbf{LI}_{n,m}$ and $\mathbf{D}_{n,m}$ are $\mathcal{D}$-compatible. In view of the above, to show that they are the six distinct maximal $\mathcal{D}$-compatible subvarieties of $\mathbf{V}_{n,m}$, $n, m > 1$, it suffices to show that they are pairwise incomparable. We use Theorem 4.1 of [5], Lemma 3.1 and Theorem 3.4. As in the proof of the Theorem 3.11, we have that $P \times P^*$ belongs to $\mathbf{LAp}_n \cap \mathbf{RAp}_n$ but not to the other five, $P^* \times RZ_2^1$ is in $\mathbf{R}_{n,m} \cap \mathbf{LI}_{n,m}$ but not in the other five, $P \times LZ_2^1$ is in $\mathbf{L}_{n,m} \cap \mathbf{RI}_{n,m}$ but not in the other five, and $B \times B^*$ is in $\mathbf{D}_{n,m}$ but not in the other five. Further, if p is a prime divisor of m, then $P^* \times C \times C_p$ is in $\mathbf{RC}_{n,m} \cap \mathbf{LI}_{n,m}$ but not in the other five while $P \times C \times C_p$ is in $\mathbf{LC}_{n,m} \cap \mathbf{RI}_{n,m}$ but not in the other five. □

Remark. The only case which is missing from Theorems 3.11 and 3.12 is the case $\mathbf{V}_{1,m}$. Here $\mathbf{V}_{1,m} = \mathbf{D}_{1,m}$ consists of completely regular semigroups only, and so $\mathbf{V}_{1,m}$ is itself $\mathcal{D}$-compatible.

Corollary 3.13. *Let $\mathbf{V}$ be a subvariety of $\mathbf{V}_{n,m}$. Then $\mathbf{V} \cap \mathbf{LAp}_n \cap \mathbf{RAp}_n$, $\mathbf{V} \cap \mathbf{LC}_{n,m} \cap \mathbf{RI}_{n,m}$, $\mathbf{V} \cap \mathbf{RC}_{n,m} \cap \mathbf{LI}_{n,m}$, $\mathbf{V} \cap \mathbf{L}_{n,m} \cap \mathbf{RI}_{n,m}$, $\mathbf{V} \cap \mathbf{R}_{n,m} \cap \mathbf{LI}_{n,m}$ and $\mathbf{V} \cap \mathbf{D}_{n,m}$ are $\mathcal{D}$-compatible subvarieties of $\mathbf{V}$ and every $\mathcal{D}$-compatible subvariety of $\mathbf{V}$ is contained in one of these.*

Corollary 3.14. *Let $\mathbf{V} \subseteq \mathbf{V}_{n,m}$. Then $\mathbf{V}$ is $\mathcal{D}$-compatible if $\mathbf{V}$ is an $\mathcal{R}$-compatible subvariety of $\mathbf{LI}_{n,m}$ or an $\mathcal{L}$-compatible subvariety of $\mathbf{RI}_{n,m}$ or a subvariety of $\mathbf{D}_{n,m}$.*

Proof. The proof follows immediately from Theorem 4.6 of [5] and its dual, Corollary 3.2 and Theorems 3.11 and 3.12. $\square$

Remark. Let S be a periodic semigroup of bounded index and period, that is, $S \in \mathbf{V}_{n,m}$ for some m, n. Then by Theorem 3.4 and Corollary 3.14, S generates a $\mathcal{D}$-compatible variety if and only if $S \in \mathbf{LI}_{n,m}$ and S generates an $\mathcal{R}$-compatible variety, or, $S \in \mathbf{RI}_{n,m}$ and S generates an $\mathcal{L}$-compatible variety, or $Gr(S)$ is an ideal of S. In view of Corollary 4.8 of [5] and its dual and Lemma 3.1 the above translates into necessary and sufficient conditions which can be formulated in terms of the structure of S. In particular, if S is a finite semigroup, then it is decidable whether S generates a $\mathcal{D}$-compatible variety.

The intersection of an $\mathcal{L}$-compatible variety and an $\mathcal{R}$-compatible variety yields a variety which is $\mathcal{K}$-compatible for each of the Green relations $\mathcal{K} = \mathcal{H}, \mathcal{L}, \mathcal{R}, \mathcal{D}, \mathcal{J}$ (for the overcommutative case, see Theorem 6.5 of [4]). Conversely, every semigroup variety which is compatible for each of the Green relations is in a trivial way the intersection of an $\mathcal{L}$-compatible and an $\mathcal{R}$-compatible variety. In the following we investigate the subvarieties of $\mathbf{V}_{n,m}$ which are maximal for being compatible for each of the Green relations.

Theorem 3.15.

(i) $\mathbf{L}_{1,m} \cap \mathbf{R}_{1,m}$ *is the greatest subvariety of* $\mathbf{V}_{1,m}$ *which is compatible for each of the Green relations.*

(ii) *For $n > 1$, $\mathbf{L}_{n,1} \cap \mathbf{R}_{n,1}$, $\mathbf{L}_{n,1} \cap \mathbf{RAp}_n$, $\mathbf{LAp}_n \cap \mathbf{R}_{n,1}$ and $\mathbf{LAp}_n \cap \mathbf{RAp}_n$ are the four distinct subvarieties of $\mathbf{V}_{n,1}$ which are maximal for being compatible for each of the Green relations, and every subvariety of $\mathbf{V}_{n,1}$ which is compatible for each of the Green relations is contained in such a maximal one.*

(iii) *For $n > 1$, $m > 1$, $\mathbf{L}_{n,m} \cap \mathbf{R}_{n,m}$, $\mathbf{L}_{n,m} \cap \mathbf{RAp}_n$, $\mathbf{LAp}_n \cap \mathbf{R}_{n,m}$, $\mathbf{LAp}_n \cap \mathbf{RAp}_n$ and $\mathbf{LC}_{n,m} \cap \mathbf{RC}_{n,m}$ are the five distinct subvarieties of $\mathbf{V}_{n,m}$ which are maximal for being compatible for each of the Green relations, and every subvariety of $\mathbf{V}_{n,m}$ which is compatible for each of the Green relations is contained in such a maximal one.*

Proof. (i) $\mathbf{V}_{1,m} = \mathbf{D}_{1,m}$ consists of completely regular semigroups only. $S \in \mathbf{V}_{1,m}$ is compatible for each of the Green relations if and only if S is a regular band of groups, that is, if and only if $S \in \mathbf{L}_{1,m} \cap \mathbf{R}_{1,m}$.

(ii) Let $n > 1$. Each of the four varieties mentioned in statement (ii) is compatible for each of the Green relations, since each of these four is, by Theorem 4.1 of [5] and its dual, the intersection of an $\mathcal{L}$-compatible and an $\mathcal{R}$-compatible subvariety of $\mathbf{V}_{n,1}$. In order to show that these varieties are pairwise incomparable it suffices to find for each variety among these four a semigroup which belongs to this variety but not to the other three. Four semigroups that satisfy this requirement are $LZ_2^1 \times RZ_2^1 \in \mathbf{L}_{n,1} \cap \mathbf{R}_{n,1}$, $P \times LZ_2^1 \in \mathbf{L}_{n,1} \cap \mathbf{RAp}_n$, $P^* \times RZ_2^1 \in \mathbf{LAp}_n \cap \mathbf{R}_{n,1}$ and $P \times P^* \in \mathbf{LAp}_n \times \mathbf{RAp}_n$.

If $\mathbf{V} \subseteq \mathbf{V}_{n,1}$ is compatible for each of the Green relations, then $\mathbf{V}$ is contained in a maximal $\mathcal{L}$-compatible and in a maximal $\mathcal{R}$-compatible subvariety of $\mathbf{V}_{n,1}$. The result thus follows from Theorem 4.6 of [5] and its dual.

(iii) Let $n, m > 1$. Each of the five varieties mentioned in statement (iii) is compatible for each of the Green relations, since each of these five is, by Theorem 4.1 of [5] and its dual, the intersection of an $\mathcal{L}$-compatible and an $\mathcal{R}$-compatible subvariety of $\mathbf{V}_{n,m}$. Using the same four semigroups as in the proof of (ii) we see that $\mathbf{L}_{n,m} \cap \mathbf{R}_{n,m}$, $\mathbf{L}_{n,m} \cap \mathbf{RAp}_n$, $\mathbf{LAp}_n \cap \mathbf{R}_{n,m}$ and $\mathbf{LAp}_n \cap \mathbf{RAp}_n$ are four pairwise incomparable subvarieties of $\mathbf{V}_{n,m}$ and that each of them is not contained in $\mathbf{LC}_{n,m} \cap \mathbf{RC}_{n,m}$. On the other hand, if p is a prime divisor of m, then $C_p \times C \in \mathbf{LC}_{n,m} \cap \mathbf{RC}_{n,m}$ but $C_p \times C$ does not belong to any of the other four varieties listed. Therefore the five listed varieties are pairwise incomparable.

Let $\mathbf{V} \subseteq \mathbf{V}_{n,m}$ be any variety which is compatible for each of the Green relations. Again $\mathbf{V}$ is contained in a maximal $\mathcal{L}$-compatible subvariety of $\mathbf{V}_{n,m}$ and a maximal $\mathcal{R}$-compatible subvariety of $\mathbf{V}_{n,m}$. In view of Theorem 4.6 of [5] and its dual, in order to show that $\mathbf{V}$ is contained in one of the five listed varieties, it suffices to prove that $\mathbf{LC}_{n,m} \cap \mathbf{RAp}_n$ and $\mathbf{LAp}_n \cap \mathbf{RC}_{n,m}$ are contained in $\mathbf{LAp}_n \cap \mathbf{RAp}_n$, and $\mathbf{L}_{n,m} \cap \mathbf{RC}_{n,m}$ and $\mathbf{LC}_{n,m} \cap \mathbf{R}_{n,m}$ are contained in $\mathbf{L}_{n,m} \cap \mathbf{R}_{n,m}$.

From Theorem 4.1(ii) and Corollary 4.2 of [5] it follows that $\mathbf{RC}_{n,1} \subseteq \mathbf{RAp}_n$. Therefore, since $\mathbf{LAp}_n$ is aperiodic,

$$\begin{aligned}\mathbf{LAp}_n \cap \mathbf{RC}_{n,m} &= \mathbf{LAp}_n \cap (\mathbf{V}_{n,1} \cap \mathbf{RC}_{n,m}) \\ &= \mathbf{LAp}_n \cap \mathbf{RC}_{n,1} \\ &\subseteq \mathbf{LAp}_n \cap \mathbf{RAp}_n.\end{aligned}$$

Dually, $\mathbf{LC}_{n,m} \cap \mathbf{RAp}_n \subseteq \mathbf{LAp}_n \cap \mathbf{RAp}_n$.

From Theorem 4.1 of [5] and its dual we have that $\mathbf{L}_{n,m} \cap \mathbf{RC}_{n,m}$ contains neither P, P^* nor C. Thus, if $S \in \mathbf{L}_{n,m} \cap \mathbf{RC}_{n,m}$, then $Gr(S)$ is an ideal of S by Lemma 2.5 of [4], whereas $Gr(S)$ is both an $\mathcal{L}$-compatible and an $\mathcal{R}$-compatible completely regular semigroup. Therefore by Theorem 4.1(i) of [5] and its dual, $\mathbf{L}_{n,m} \cap \mathbf{RC}_{n,m} \subseteq \mathbf{L}_{n,m} \cap \mathbf{R}_{n,m}$. That $\mathbf{LC}_{n,m} \cap \mathbf{R}_{n,m} \subseteq \mathbf{L}_{n,m} \cap \mathbf{R}_{n,m}$ follows in a dual way. $\square$

Remark. Comparing Theorems 3.11 and 3.12 with Theorem 3.15 we see that in some cases the intersection of a maximal $\mathcal{L}$-compatible subvariety of $\mathbf{V}_{n,m}$ with a maximal $\mathcal{R}$-compatible subvariety of $\mathbf{V}_{n,m}$ yields a maximal $\mathcal{D}$-compatible subvariety of $\mathbf{V}_{n,m}$. For $n > 1$ and $m = 1$ this will be the case precisely with $\mathbf{LAp}_n \cap \mathbf{RAp}_n$, $\mathbf{LAp}_n \cap \mathbf{R}_{n,1}$ and $\mathbf{L}_{n,1} \cap \mathbf{RAp}_n$, whereas for $n, m > 1$ this happens only with $\mathbf{LAp}_n \cap \mathbf{RAp}_n$.

4. Periodic varieties minimal for not being $\mathcal{D}$-compatible

In this section we find the periodic semigroup varieties which are minimal for not being $\mathcal{D}$-compatible. In the next section we shall see that these constitute in fact the only varieties minimal for not being $\mathcal{D}$-compatible. Among the varieties that we shall discover there are two infinite families, each indexed by the set of prime numbers. When we restrict ourselves to aperiodic varieties however we have the following.

Theorem 4.1. *The varieties* $\mathbf{HSP}(Q)$, $\mathbf{HSP}(Q^*)$ *and* $\mathbf{HSP}(B_2)$ *are the three distinct aperiodic varieties that are minimal for not being* $\mathcal{D}$*-compatible. Every aperiodic variety which is not* $\mathcal{D}$*-compatible contains one of these.*

Proof. Clearly $\mathbf{HSP}(Q)$, $\mathbf{HSP}(Q^*)$ and $\mathbf{HSP}(B_2)$ are aperiodic and not $\mathcal{D}$-compatible. Using Lemma 2.10 of [5] and its dual we see that $RZ_2 \in \mathbf{HSP}(Q)$ but $RZ_2 \notin \mathbf{HSP}(Q^*)$. By symmetry, and using Lemma 2.12 of [5] and its dual, we see that $\mathbf{HSP}(Q)$, $\mathbf{HSP}(Q^*)$ and $\mathbf{HSP}(B_2)$ are pairwise incomparable varieties. We complete the proof by showing that if an aperiodic variety contains neither Q, Q^* nor B_2 then it must be a $\mathcal{D}$-compatible variety.

Let $\mathbf{V} \subseteq \mathbf{V}_{n,1}$ be an aperiodic variety which contains neither Q, Q^* nor B_2. If $\mathbf{V}$ contains neither P, P^* nor C then $\mathbf{V} \subseteq \mathbf{D}_{n,1}$ by Lemma 2.5 of [4] and Theorem 3.4, and so $\mathbf{V}$ is $\mathcal{D}$-compatible by Theorem 3.7. If $\mathbf{V}$ contains C but not P nor P^*, then by Lemma 2.13(i) of [5] and its dual, $\mathbf{V}$ cannot contain LZ_2 nor RZ_2. In this case $\mathbf{V}$ is again $\mathcal{D}$-compatible since in view of Lemma 3.1(i) of [5] and its dual, $\mathbf{V}$ consists of semigroups for which the $\mathcal{D}$-relation is trivial. By symmetry it suffices to investigate the case where $\mathbf{V}$ contains P^*. By the dual of Lemma 4.4 of [5] we have that $\mathbf{V} \subseteq \mathbf{LAp}_n$. If $\mathbf{V}$ also contains P, then by symmetry $\mathbf{V} \subseteq \mathbf{LAp}_n \cap \mathbf{RAp}_n$ and $\mathbf{V}$ is $\mathcal{D}$-compatible by Theorem 3.7. We shall henceforth assume that $\mathbf{V}$ contains P^* but not P. If $C \in \mathbf{V}$ then $RZ_2 \notin \mathbf{V}$ by Lemma 2.13(i) of [5], and so by Lemma 3.1(i) of [5], $\mathbf{V}$ consists of semigroups for which the $\mathcal{R}$-relation is trivial. Consequently by Theorem 4.1(ii) of [5], $\mathbf{V} \subseteq \mathbf{LAp}_n \cap \mathbf{RAp}_n$, and again $\mathbf{V}$ is $\mathcal{D}$-compatible by Theorem 3.7. Finally, assume that $\mathbf{V}$ contains P^* but not P nor C. For $S \in \mathbf{V}$, $Gr(S)$ is a right ideal of S by Lemma 2.14 of [5]. Since $\mathbf{V} \subseteq \mathbf{LAp}_n$, and $B \notin \mathbf{LAp}_n$ by the dual of Theorem 4.1(ii) of [5], it follows that the band $Gr(S)$ is $\mathcal{R}$-compatible by Lemma 2.8 of [5]. By Theorem 4.1(i) of [5] we thus have that $\mathbf{V} \subseteq \mathbf{R}_{n,1}$. Thus $\mathbf{V} \subseteq \mathbf{LAp}_n \cap \mathbf{R}_{n,1}$ and again $\mathbf{V}$ is $\mathcal{D}$-compatible by Theorem 3.11. □

We set out to find the remaining periodic varieties that are minimal for not being $\mathcal{D}$-compatible.

Lemma 4.2.

(i) *Let $\mathbf{V}$ be a subvariety of $\mathbf{V}_{n,m}$ which does not contain P nor P^*. Then $\mathbf{V}$ is not $\mathcal{D}$-compatible if and only if $m > 1$ and $\mathbf{V}$ contains $L_1(p)$ for some prime divisor p of m.*
(ii) *The varieties $\mathbf{HSP}(L_1(p))$, p prime, are minimal for not being $\mathcal{D}$-compatible.*

Proof. (i) Clearly a variety containing a semigroup of the form $L_1(p)$, p a prime, is not $\mathcal{D}$-compatible by Lemma 2.2.

Assume that $\mathbf{V} \subseteq \mathbf{V}_{n,m}$ such that neither P nor P^* belong to $\mathbf{V}$ and such that $\mathbf{V}$ is not $\mathcal{D}$-compatible. Since P divides Q, P^* divides Q^* and both P and P^* divide B_2, $\mathbf{V}$ cannot contain Q, Q^* nor B_2. Therefore by Lemma 4.1, $m > 1$. To complete the proof we need to show that if $\mathbf{V}$ is a periodic semigroup variety which contains neither P nor P^* nor the semigroups $L_1(p)$, p prime, then $\mathbf{V}$ is $\mathcal{D}$-compatible.

Let $\mathbf{V} \subseteq \mathbf{V}_{n,m}$ such that $\mathbf{V}$ contains neither P, nor P^* nor $L_1(p)$ for any prime p. If $C \notin \mathbf{V}$, then $\mathbf{V} \subseteq \mathbf{D}_{n,m}$ by Lemma 2.5 of [4] and Theorem 3.4, and so $\mathbf{V}$ is $\mathcal{D}$-compatible by Theorem 3.7. We shall henceforth assume that $\mathbf{V}$ contains C. Since P and P^* are not in $\mathbf{V}$ we have that LZ_2 and RZ_2 are not in $\mathbf{V}$ by Lemma 2.13(i) of [5] and its dual. A fortiori $L(p)$ ad $R(p)$ are not in $\mathbf{V}$ for any prime p, and also not the bands B and B^*. Since for any prime p, P and P^* are homomorphic images of $L_2(p)$ and $R_2(p)$, respectively, $\mathbf{V}$ cannot contain the semigroups $L_2(p)$ nor $R_2(p)$, p any prime. Therefore by Lemma 2.6 of [5] and Theorem 3.6 of [5] and its dual we conclude that $\mathbf{V}$ is both $\mathcal{L}$-compatible and $\mathcal{R}$-compatible, and thus $\mathcal{D}$-compatible.

(ii) Let p be any prime. By Lemma 2.6 of [5] and Theorem 3.6 of [5] and its dual we have that every proper subvariety of $\mathbf{HSP}(L_1(p))$ is both $\mathcal{L}$-compatible and $\mathcal{R}$-compatible and thus $\mathcal{D}$-compatible. □

We now arrive at the main theorem of this section.

Theorem 4.3. *The periodic semigroup varieties which are minimal for not being $\mathcal{D}$-compatible are the varieties each generated by one of the semigroups*

(i) B_2, Q, Q^*,
(ii) $L_1(p)$, *p prime,*
(iii) $P^* \times C_p \times P$, *p prime.*

Every periodic semigroup variety which is not $\mathcal{D}$-compatible contains such a minimal one.

Proof. Let $\mathbf{V}$ be a periodic semigroup variety which contains neither one of the semigroups listed in (i), (ii) or (iii). We want to show that $\mathbf{V}$ is $\mathcal{D}$-compatible. By Theorem 4.1 it suffices to assume that $C_p \in \mathbf{V}$ for some prime p, and by Lemma 4.2 and symmetry we may as well also assume that $P^* \in \mathbf{V}$. By the dual of Corollary 3.3 we then have that $B \notin \mathbf{V}$ and $L(q) \notin \mathbf{V}$ for any prime q. Also $P \notin \mathbf{V}$ since $P^* \times C_p \in \mathbf{V}$ but $P^* \times C_p \times P \notin \mathbf{V}$. A fortiori $L_2(q) \notin \mathbf{V}$ for any prime q since P is a homomorphic image of such a $L_2(q)$. By Theorem 3.6 of [5] it follows that $\mathbf{V}$ is $\mathcal{R}$-compatible. Since $\mathbf{V}$ contains P^* but not Q^* nor B_2, $\mathbf{V} \subseteq \mathbf{LI}_{n,m}$ for some n, m, by the dual of Lemma 3.1(ii). Since $\mathbf{V}$ is $\mathcal{R}$-compatible and is contained in $\mathbf{LI}_{n,m}$ for some n, m, it follows from Lemma 3.6 that $\mathbf{V}$ is indeed $\mathcal{D}$-compatible. We proved that every periodic semigroup variety which is not $\mathcal{D}$-compatible contains one of the semigroups listed in (i)–(iii).

We know that the varieties, each generated by the semigroups listed in (i) and (ii) are minimal for not being $\mathcal{D}$-compatible by Theorem 4.1 and Lemma 4.2. Also the varieties $\mathbf{HSP}(P^* \times C_p \times P)$, p prime, are not $\mathcal{D}$-compatible by Lemma 2.1. To complete the proof it suffices to show that $\mathbf{HSP}(P^* \times C_p \times P)$, p prime, does not contain any of the semigroups listed in (i) and (ii). We have that $P^* \times C_p \times P$ satisfies the identity $x^p y^p \approx y^p x^p$ while Q and Q^* do not, and $P^* \times C_p \times P$ satisfies $(xyx)^{p+1} \approx xyx$ while B_2 does not. Also $P^* \times C_p \times P$ satisfies $x^p z y^{p+1} \approx y^{p+1} z x^p$, while $L_1(q)$ does not satisfy this identity for any prime q by Lemma 2.1 of [4]. □

Corollary 4.4. *Let $\mathbf{V}$ be a subvariety of the variety $\mathbf{V}_{n,m}$ and let q be the product of the primes which occur in the prime factorization of m. Then the following are equivalent*:

(i) $\mathbf{V}$ *is $\mathcal{D}$-compatible,*
(ii) $\mathbf{V} \cap \mathbf{V}_{2,q}$ *is $\mathcal{D}$-compatible,*
(iii) $\mathbf{V}$ *does not contain any of the semigroups Q, Q^* nor B_2 and also not the semigroups $L_1(p)$, $P^* \times C_p \times P$, p any prime divisor of m.*

5. Pseudovarieties

We shall give the analogues for pseudovarieties of the main results obtained in Sections 3 and 4. The technique of proof is precisely as in Section 5 of [4] and Section 5 of [5], and so we shall omit the details here. For all undefined notation and terminology we refer to [4,5].

Theorem 5.1. *The semigroup pseudovarieties which are minimal for not being $\mathcal{D}$-compatible are the pseudovarieties each generated by one of the semigroups Q, Q^* or B_2 or by one of the semigroups $L_1(p)$, $P^* \times C_p \times P$, p a prime. Each semigroup pseudovariety which is not $\mathcal{D}$-compatible contains such a minimal one.*

Proof. Immediate from Theorem 4.3. □

Using the remark which follows Corollary 3.14 we have

Theorem 5.2. *A finite semigroup generates a $\mathcal{D}$-compatible pseudovariety if and only if it generates a $\mathcal{D}$-compatible variety. There exists an algorithm which decides whether a given finite semigroup generates a $\mathcal{D}$-compatible pseudovariety or not.*

In [5] we introduced the pseudovarieties $\mathbf{R} = \mathbf{Fin}(\bigcup_{n,m\geqslant 1} \mathbf{R}_{n,m})$, $\mathbf{RAp} = \mathbf{Fin}(\bigcup_{n\geqslant 1} \mathbf{RAp}_n)$ and $\mathbf{RC} = \mathbf{Fin}(\bigcup_{n,m\geqslant 1} \mathbf{RC}_{n,m})$. The pseudovarieties $\mathbf{L}, \mathbf{LAp}$ and $\mathbf{LC}$ are the left–right duals of $\mathbf{R}, \mathbf{RAp}$ and $\mathbf{RC}$, respectively. Since the varieties $\mathbf{RI}_{n,m}$, $n, m \geqslant 1$, form an up-directed poset we have that $\bigcup_{n,m\geqslant 1} \mathbf{RI}_{n,m}$ is generalized variety and $\mathbf{RI} = \mathbf{Fin}(\bigcup_{n,m\geqslant 1} \mathbf{RI}_{n,m})$ a pseudovariety. The left–right dual of $\mathbf{RI}$ will be denoted by $\mathbf{LI}$. Similarly $\mathbf{D} = \mathbf{Fin}(\bigcup_{n,m\geqslant 1} \mathbf{D}_{n,m})$ is a pseudovariety.

Theorem 5.3. *There are six distinct maximal $\mathcal{D}$-compatible pseudovarieties, namely* $\mathbf{LAp} \cap \mathbf{RAp}$, $\mathbf{LC} \cap \mathbf{RI}$, $\mathbf{RC} \cap \mathbf{LI}$, $\mathbf{L} \cap \mathbf{RI}$, $\mathbf{R} \cap \mathbf{LI}$ *and* $\mathbf{D}$. *Every $\mathcal{D}$-compatible pseudovariety is contained in one of these.*

Proof. Immediate from Theorem 3.12. □

From Theorem 3.15 we derive the following. Recall that we use the notation $\mathbf{C} = \mathbf{LC} \cap \mathbf{RC}$.

Theorem 5.4. *There are five distinct pseudovarieties which are maximal for being compatible for each of the Green relations, namely* $\mathbf{L} \cap \mathbf{R}$, $\mathbf{L} \cap \mathbf{RAp}$, $\mathbf{LAp} \cap \mathbf{R}$, $\mathbf{LAp} \cap \mathbf{RAp}$ *and* $\mathbf{C}$. *Every pseudovariety which is compatible for each of the Green relations is contained in such a maximal one.*

6. $\mathcal{D}$-Compatible overcommutative semigroup varieties

In [4,5] we investigated the varieties $\mathbf{RC}_k$, $k > 0$. For $k > 1$, $\mathbf{RC}_k$ is determined by the identities

$$y^{2k}x \approx y^{k+1}xy^{k-1} \approx y^k xy^k \approx y^{k-1}xy^{k+1}, \tag{9}$$

whereas $\mathbf{RC}_1$ is determined by

$$y^2x \approx yxy \approx xy^2. \tag{10}$$

We proved that the varieties $\mathbf{RC}_k, k > 0$, are $\mathcal{R}$-compatible and conversely, any $\mathcal{R}$-compatible overcommutative variety is contained in such an $\mathbf{RC}_k$ for k sufficiently large (see [5, Theorem 6.4]). For $k > 0$ the left–right dual of $\mathbf{RC}_k$ will be denoted by $\mathbf{LC}_k$. For $k > 0$ we let $\mathbf{LI}_k$ be the variety determined by

$$zx^k y^k \approx zy^k x^k. \tag{11}$$

The left–right dual of $\mathbf{LI}_k$ will be denoted by $\mathbf{RI}_k$. Clearly for all $k > 0$, the varieties $\mathbf{LC}_k, \mathbf{RC}_k, \mathbf{LI}_k$ and $\mathbf{RI}_k$ are overcommutative since the identities that determine them are balanced.

We start off with a few technical results concerning the semigroups that belong to $\mathbf{RC}_k$.

Lemma 6.1. *Let $k > 0$, $S \in \mathbf{RC}_k$ and $a \in S$.*

(i) *$T = \{s \in S^1 \mid tsa = a$ for some $t \in S^1\}$ is a subsemigroup of S and Ta is the $\mathcal{L}$-class of S containing a, whereas aT is contained in the $\mathcal{R}$-class of S containing a.*
(ii) *If $tsa = a$ for some $s, t \in S^1$ then $a = ast = ats$.*

Proof. Assume that $wva = a = tsa$ for some $s, t, v, w \in S^1$. Since S satisfies the identities (9), we have

$$a = tsa = (ts)^{2k} a = (ts)^{2k} wva = (ts)^k w (ts)^k va,$$

whence $sv \in T$. Therefore T is subsemigroup of S. Clearly Ta is the $\mathcal{L}$-class of S containing a.

Again, assume that $a = tsa$ for some $s, t \in S^1$. Then, since S satisfies the identities (9), we have

$$\begin{aligned} a = tsa = (ts)^{2k} a = (ts)^k a (ts)^k = a(ts)^k \\ = (ts)^{k-1} a (ts)^{k+1} = a(ts)^{k+1} \end{aligned}$$

and therefore

$$ats = a(ts)^k ts = a(ts)^{k+1} = a.$$

Since by Theorem 6.2 of [5] S satisfies the implication

$$x = xyz \quad \Rightarrow \quad xyu = xuy$$

we also have $a = ast$. We proved (ii). That aT is contained in the $\mathcal{R}$-class of a now follows immediately from (ii). $\square$

Theorem 6.2. *If $\mathbf{V}$ is an overcommutative $\mathcal{D}$-compatible variety then $P \notin V^*$ or $P \notin \mathbf{V}$. For an overcommutative variety $\mathbf{V}$ the following are equivalent:*

(i) *$\mathbf{V}$ is $\mathcal{D}$-compatible and $P \notin \mathbf{V}$,*
(ii) *$\mathbf{V} \subseteq \mathbf{RC}_k \cap \mathbf{LI}_k$ for some $k > 0$,*
(iii) *$\mathbf{V}$ does not contain P, Q^* and $L_1(p)$, p prime,*
(iv) *$\mathbf{V} \cap \mathbf{V}_{n,m} \subseteq \mathbf{RC}_{n,m} \cap \mathbf{LI}_{n,m}$ for all n, m,*
(v) *the periodic subvarieties of $\mathbf{V}$ are $\mathcal{D}$-compatible and $P \notin \mathbf{V}$,*

(vi) **V** *is contained in the quasivariety determined by the implications*

$$x = xyz \quad \Rightarrow \quad xyu = xuy, \tag{12}$$

$$xyz = z \quad \Rightarrow \quad uyz = uzy. \tag{13}$$

Proof. If **V** is an overcommutative $\mathcal{D}$-compatible variety then **V** contains C_p for every prime p. Since **V** cannot contain $P^* \times C_p \times P$, p prime, by Lemma 2.1, it follows that $P^* \notin \mathbf{V}$ or $P \notin \mathbf{V}$.

(i) ⇒ (iii). Assume that **V** is $\mathcal{D}$-compatible and $P \notin \mathbf{V}$. By Lemma 2.2, **V** cannot contain $L_1(p)$, p prime. Obviously **V** cannot contain Q^* since Q^* is not $\mathcal{D}$-compatible.

(iii) ⇒ (ii). Assume that **V** does not contain P, Q^* and $L_1(p)$, p prime. Therefore $\mathbf{V} \subseteq \mathbf{RC}_\ell$ for some $\ell > 0$ by Theorem 6.2 of [5]. Since **V** does not contain Q^*, **V** satisfies a balanced identity $u \approx v$ which is not satisfied by Q^*. Using the dual of Lemma 2.10 of [5], we have that either u and v begin with distinct variables x and y, respectively, or otherwise, for distinct variables x, y and z, u and v begin with zx and zy, respectively, and $n_z(u) = n_z(v) = 1$. Obviously an identity of the former kind has an identity of the second kind as a consequence. We shall therefore henceforth assume that **V** satisfies a balanced identity $u \approx v$ of the second kind. In this identity we substitute z by z, x by x^ℓ and all other variables by y^ℓ. The resulting identity is of the form

$$zp\bigl(x^\ell, y^\ell\bigr) \approx zq\bigl(x^\ell, y^\ell\bigr) \quad \text{where } p(x,y) \approx q(x,y)$$

is a balanced identity involving x and y only, and where $p(x,y)$ and $q(x,y)$ begin with x and y, respectively. Since $\mathbf{V} \subseteq \mathbf{RC}_\ell$, we may apply the identity $y^{2\ell}x \approx y^\ell x y^\ell$ repeatedly and from $zp(x^\ell, y^\ell) \approx zq(x^\ell, y^\ell)$ derive a consequence of the form $zx^{s\ell}y^{t\ell} \approx zy^{t\ell}x^{s\ell}$, $s, t > 0$. Putting $k = st\ell$ and substituting in the last identity z by z, x by x^t and y by y^s, we find the consequence $zx^k y^k \approx zy^k x^k$ which is satisfied in **V**. In other words, $\mathbf{V} \subseteq \mathbf{LI}_k$. Since $\ell \leqslant k$ we have $\mathbf{RC}_\ell \subseteq \mathbf{RC}_k$ by Proposition 6.1(ii) of [5]. Therefore $\mathbf{V} \subseteq \mathbf{RC}_k \cap \mathbf{LI}_k$.

(ii) ⇒ (vi). Let $\mathbf{V} \subseteq \mathbf{RC}_k \cap \mathbf{LI}_k$ for some $k > 0$. Then **V** satisfies the implication (12) by Theorem 6.2 of [5]. Assume that $S \in \mathbf{V}$ and let $s, t, a \in S$ such that $tsa = a$. In order to show that S satisfies the implication (13), we let c be any element of S and we need to prove that $csa = cas$.

Since $S \in \mathbf{RC}_k$ we have

$$s^{2k}a = s^{k-1}as^{k+1} = s^k a s^k. \tag{14}$$

Also $as^{k+1}t^k = as$ and $as^k t^k = a$ by Lemma 6.1 and since S satisfies the implication (12). Therefore (14) yields

$$s^{k-1}as = s^{k-1}as^{k+1}t^k = s^k a s^k t^k = s^k a.$$

This in particular implies that

$$s^{k-1+\ell}a = s^{k-1}as^\ell \quad \text{for all } \ell \geqslant 1. \tag{15}$$

From Lemma 6.1 we know that there exists $w \in S$ such that $ws^k a = a$. Then

$$\begin{aligned}
(sws^k)^k a &= s(ws^k)(sws^k)^{k-1} a \\
&= s(ws^{k+1})^{k-1} ws^k a \\
&= s(ws^{k+1})^{k-1} a \\
&= s(ws^{k+1})^{k-2} ws^{k+1} a \\
&= s(ws^{k+1})^{k-2} ws^k as \quad \text{(by (15))} \\
&= s(ws^{k+1})^{k-2} as \\
&= \cdots \\
&= sas^{k-1},
\end{aligned}$$

and so

$$\begin{aligned}
(ws^k)^k (sws^k)^k a &= (ws^k)^k sas^{k-1} \\
&= (ws^k)^{k-1} ws^{k+1} as^{k-1} \\
&= (ws^k)^k as^k \\
&= as^k.
\end{aligned}$$

Therefore, and in view of the fact that $S \in \mathbf{LI}_k$,

$$\begin{aligned}
cas^k &= c(ws^k)^k (sws^k)^k a \\
&= c(sws^k)^k (ws^k)^k a \\
&= c(sws^k)^k a \\
&= csas^{k-1}.
\end{aligned}$$

Again, $as^k t^{k-1} = as$ and $as^{k-1} t^{k-1} = a$ by Lemma 6.1 and since S satisfies the implication (12). Thus

$$cas = cas^k t^{k-1} = csas^{k-1} t^{k-1} = csa,$$

as required.

(vi) $\Rightarrow$ (i). Let $\mathbf{V}$ be contained in the quasivariety which is determined by (12) and (13). Since $\mathbf{V}$ is in the quasivariety determined by (12), we know that $\mathbf{V}$ is $\mathcal{R}$-compatible and $\mathbf{V} \subseteq \mathbf{RC}_k$ for some $k > 0$ by Theorem 6.2 of [5]. Let $S \in \mathbf{V}$ and assume that a and b are distinct $\mathcal{L}$-related elements of S. Then $b = sa$ and $a = tb$ for some $s, t \in S$. Therefore $a = tsa$ and by Lemma 6.1 we have that $a \mathcal{R} as$ in S. Let c be any element of S. Since S satisfies (13), we have

$$cb = csa = cas \mathcal{R} ca.$$

Since S is $\mathcal{R}$-compatible, we have from Lemma 3.5 that S is $\mathcal{D}$-compatible. Therefore $\mathbf{V}$ is $\mathcal{D}$-compatible. The semigroup P does not satisfy (13) because $eea = a$ while $eea \neq eae$. Therefore $\mathbf{V}$ does not contain P.

(i) $\Rightarrow$ (iv) $\Rightarrow$ (v) by Theorems 3.7 and 3.12(i), and since $P \notin \mathbf{RC}_{n,m}$.

(v) $\Rightarrow$ (iii) by Lemma 2.2 and since Q^* is not $\mathcal{D}$-compatible. $\square$

Remark. For $n > 1$, $\mathbf{RC}_{n,m} \cap \mathbf{LI}_{n,m}$ is properly contained in $\mathbf{RC}_{n,m}$ since Q^* belongs to $\mathbf{RC}_{n,m}$ but not to $\mathbf{LI}_{n,m}$. Similarly, Q^* belongs to $\mathbf{RC}_k$ for $k > 0$ but not to $\mathbf{LI}_k$, and so for $k > 0$, $\mathbf{RC}_k \cap \mathbf{LI}_k$ is properly contained in $\mathbf{RC}_k$.

Corollary 6.3. *Let $\mathbf{V}$ be an overcommutative variety which is $\mathcal{J}$-compatible. Then $\mathbf{V}$ is $\mathcal{D}$-compatible.*

Proof. $\mathbf{V} \cap \mathbf{V}_{n,m}$ is $\mathcal{D}$-compatible for all n, m, and so $P \notin \mathbf{V}$ or $P^* \notin \mathbf{V}$ by Lemma 2.1. By Theorem 6.3 and its dual, $\mathbf{V}$ is $\mathcal{D}$-compatible. $\square$

Corollary 6.4. *The semigroup varieties which are minimal for not being $\mathcal{D}$-compatible are the periodic semigroup varieties listed in Theorem* 4.3 *and each semigroup variety which is not $\mathcal{D}$-compatible contains such a minimal one.*

Proof. The proof immediately follows from Theorems 4.3 and 6.2. $\square$

Corollary 6.5. *If $\mathbf{V}$ is a $\mathcal{D}$-compatible semigroup variety which contains a nonabelian group, then $\mathbf{V}$ is a periodic variety and for some m, n, $\mathbf{V} \subseteq \mathbf{D}_{n,m}$ or $\mathbf{V} \subseteq \mathbf{R}_{n,m} \cap \mathbf{LI}_{n,m}$ or $\mathbf{V} \subseteq \mathbf{L}_{n,m} \cap \mathbf{RI}_{n,m}$.*

Proof. The proof follows immediately from Theorems 4.1, 6.2, 6.5 of [5] and Theorems 3.7 and 6.2. $\square$

We are now in the position to determine the $\mathcal{J}$-compatible varieties. We need two auxiliary lemmas first.

Lemma 6.6. *Let S be a semigroup and ρ a congruence relation on S which is contained in the $\mathcal{J}$-relation on S. Let $T = S/\rho$. Then $a\mathcal{J}b$ in S if and only if $a\rho\mathcal{J}b\rho$ in T.*

Proof. If $a\mathcal{J}b$ then it is immediate that $a\rho\mathcal{J}b\rho$ in T. Assume that $a\rho\mathcal{J}b\rho$ in T. Then there exist $p, q, r, t \in S^1$ such that $b\rho = (r\rho)(a\rho)(t\rho)$ and $a\rho = (p\rho)(b\rho)(q\rho)$. Here of course 1ρ stands for the identity element of T^1. Therefore $b\rho rat$ and $a\rho pbq$, and thus $b\mathcal{J}rat$ and $a\mathcal{J}pbq$ in S since $\rho \subseteq \mathcal{J}$. From this we easily deduce that $a\mathcal{J}b$ in S. $\square$

Recall from Theorem 6.2 of [5] that for $k > 0$ and $S \in \mathbf{RC}_k$, the $\mathcal{R}$-relation is a congruence on S.

Lemma 6.7. *Let $k > 0$, $S \in \mathbf{RC}_k$ and $T = S/\mathcal{R}$. Then $\mathcal{J} = \mathcal{L}$ for T.*

Proof. For $a \in S$ we shall put $R_a = \bar{a}$. By Lemma 6.6 we have for $a, b \in S$ that $a\mathcal{J}b$ in S if and only if $\bar{a}\mathcal{J}\bar{b}$ in T. If this is the case, then there exist $p, q, r, t \in S^1$ such that $a = pbq$ and

$b = rat$. In particular $a = pratq$ and thus also $a = (pr)^\ell a(tq)^\ell$ for any $\ell \geqslant 1$. Therefore, and since S satisfies the identities (9),

$$\begin{aligned} a &= (pr)^{2k} a(tq)^{2k} = (pr)^k a(tq)^k (tq)^k (pr)^k \\ &= a(tq)^k (pr)^k. \end{aligned}$$

It follows that $a\mathcal{R}at$ in S, whence $\bar{a} = \bar{a}\bar{t}$. Similarly we have $\bar{b} = \bar{b}\bar{q}$. In T we thus have $\bar{a} = \bar{p}\bar{b}\bar{q} = \bar{p}\bar{b}$ and $\bar{b} = \bar{r}\bar{a}\bar{t} = \bar{r}\bar{a}$, that is, $\bar{a}\mathcal{L}\bar{b}$. □

Theorem 6.8. *A semigroup variety is $\mathcal{D}$-compatible if and only if it is $\mathcal{J}$-compatible.*

Proof. If $\mathbf{V}$ is a periodic semigroup variety then for every $S \in \mathbf{V}$ the Green relations $\mathcal{D}$ and $\mathcal{J}$ coincide and so the statement of the theorem becomes trivial in this case. We now assume that $\mathbf{V}$ is an overcommutative variety. If $\mathbf{V}$ is $\mathcal{J}$-compatible, then $\mathbf{V}$ is $\mathcal{D}$-compatible by Corollary 6.3. Conversely, assume that $\mathbf{V}$ is $\mathcal{D}$-compatible. By Theorem 6.2 we may as well assume that $\mathbf{V} \subseteq \mathbf{RC}_k$ for some $k > 0$. Let $S \in \mathbf{V}$. By Lemma 6.7 the Green relations $\mathcal{J}, \mathcal{D}$ and $\mathcal{L}$ coincide on $S/\mathcal{R}$. Thus, since $S/\mathcal{R} \in \mathbf{V}$, $\mathcal{J}$ is a congruence relation on $S/\mathcal{R}$. Using Lemma 6.6 we thus find that $\mathcal{J}$ is a congruence relation on S. Hence $\mathbf{V}$ is $\mathcal{J}$-compatible. □

Remark. It follows from Theorem 6.8 that each of the equivalent statements of Theorem 6.2 is equivalent to the statement

(vii) $\mathbf{V}$ is $\mathcal{J}$-compatible and $P \notin \mathbf{V}$.

Also, Corollaries 6.4 and 6.5 remain true if $\mathcal{D}$ is replaced by $\mathcal{J}$.

7. Monoids

We summarize in the following the analogues for varieties and pseudovarieties of monoids of the main results of Sections 3–6. The technique of proof is the same as in the corresponding sections on monoids in [4,5].

Recall that $\mathbf{S}$ denotes the variety of all semigroups and $\mathbf{M}$ the variety of all monoids. Recall that the semigroup underlying a given monoid is also called the semigroup reduct of the monoid. For every subvariety $\mathbf{V}$ of $\mathbf{S}$, $\mathbf{MV}$ denotes the subvariety of $\mathbf{M}$ consisting of the monoids whose semigroup reducts belong to $\mathbf{V}$. For any subvariety $\mathbf{V}$ of $\mathbf{S}$, $\mathbf{V}_L$ denotes the semigroup variety generated by the semigroup reducts of the members of $\mathbf{MV}$. Recall that every monoid variety is of the form $\mathbf{MV}$, $\mathbf{V}$ a subvariety of $\mathbf{S}$, and that then $\mathbf{MV} = \mathbf{MV}_L$.

Theorem 7.1. *For a semigroup variety* $\mathbf{V}$ *the following are equivalent*:

(i) $\mathbf{V}_L$ *is $\mathcal{D}$-compatible,*
(ii) $\mathbf{MV}$ *is $\mathcal{J}$-compatible,*
(iii) $\mathbf{MV}$ *is $\mathcal{D}$-compatible.*

Theorem 7.2. *The monoid varieties which are minimal for not being $\mathcal{D}$-compatible* [$\mathcal{J}$*-compatible*] *are the monoid varieties each generated by one of the monoids*:

(i) $B_2^1, Q^1, (Q^*)^1$,
(ii) $L_1(p)$, *p prime*,
(iii) $(P^* \times C_p \times P)^1$, *p prime*.

Every monoid variety which is not $\mathcal{D}$-compatible [$\mathcal{J}$-compatible] contains such a minimal one.

Proof. It suffices to show that the monoid varieties generated by each of the above listed monoids are pairwise incomparable. To show this we can use the same arguments as in the proofs of Theorems 4.1 and 4.3. $\square$

It is clear from Lemma 3.1 that the monoids whose semigroup reducts are in $\mathbf{RI}_{n,m}$ are precisely the monoids of $\mathbf{MV}_{n,m}$ for which the Green relations $\mathcal{R}$ and $\mathcal{H}$ coincide. Thus $\mathbf{MRI}_{n,m}$ is the subvariety of $\mathbf{MV}_{n,m}$ consisting of the monoids for which $\mathcal{R} = \mathcal{H}$. Using the dual of Theorem 4.1(iii) of [5] we thus have that $\mathbf{MLC}_{n,m} \cap \mathbf{MRI}_{n,m}$ consists of the monoids $M \in \mathbf{MV}_{n,m}$ for which $\mathcal{R} = \mathcal{H}$ and such that for any $a \in Gr(M)$, a is in the center of Ma. By duality and Corollary 4.2 of [5], if $M \in \mathbf{MLC}_{n,m} \cap \mathbf{MRI}_{n,m}$ or $M \in \mathbf{MRC}_{n,m} \cap \mathbf{MLI}_{n,m}$, then $Gr(M)$ is a semilattice of abelian groups. We note that $P^1 \in \mathbf{MLC}_{n,m} \cap \mathbf{MRI}_{n,m}$ but $P^1 \notin \mathbf{MRC}_{n,m} \cap \mathbf{MLI}_{n,m}$.

By Theorem 4.1(i) of [5] the monoids whose semigroup reducts belong to $\mathbf{R}_{n,m}$ are in fact completely regular monoids. Therefore $\mathbf{MR}_{n,m} \subseteq \mathbf{MV}_{1,m}$, and by duality $\mathbf{ML}_{n,m} \subseteq \mathbf{MV}_{1,m}$. Similarly, $\mathbf{MD}_{n,m} = \mathbf{MV}_{1,m}$ by Theorem 3.4.

By Theorem 4.1(ii) of [5], $\mathbf{MLAp}_n \cap \mathbf{MRAp}_n$ consists of the monoids whose $\mathcal{D}$-classes are trivial.

Using Theorems 3.11 and 3.12 we thus have the following.

Theorem 7.3.

(i) $\mathbf{MV}_{1,m}$ *is $\mathcal{D}$-compatible.*
(ii) *For $n > 1$, the varieties $\mathbf{MLAp}_n \cap \mathbf{MRAp}_n$ and $\mathbf{MV}_{1,1}$ are two distinct maximal $\mathcal{D}$-compatible subvarieties of $\mathbf{MV}_{n,1}$ and every $\mathcal{D}$-compatible subvariety of $\mathbf{MV}_{n,1}$ is contained in one of these two.*
(iii) *For $n, m > 1$, $\mathbf{MRC}_{n,m} \cap \mathbf{MLI}_{n,m}$, $\mathbf{MLC}_{n,m} \cap \mathbf{MRI}_{n,m}$ and $\mathbf{MD}_{n,m}$ are three distinct subvarieties of $\mathbf{MV}_{n,m}$ and every $\mathcal{D}$-compatible subvariety of $\mathbf{MV}_{n,m}$ is contained in one of these three.*

The following is the analogue for pseudovarieties of Theorem 7.2.

Theorem 7.4. *The pseudovarieties of monoids which are minimal for not being $\mathcal{D}$-compatible are the pseudovarieties each generated by one of the monoids listed in the statement of Theorem 7.2. Every pseudovariety of monoids which is not $\mathcal{D}$-compatible contains such a minimal one.*

Put $\mathbf{MRC} = \mathbf{Fin}(\bigcup_{m,n>1} \mathbf{MRC}_{n,m}), \mathbf{MRI} = \mathbf{Fin}(\bigcup_{n,m} \mathbf{MRI}_{n,m})$ and $\mathbf{MD} = \mathbf{Fin}(\bigcup_{n,m} \mathbf{MD}_{n,m})$. The pseudovarieties **MLC** and **MLI** are the left–right duals of **MRC** and **MRI**, respectively.

Theorem 7.5. *The pseudovarieties* $\mathbf{MRC} \cap \mathbf{MLI}$, $\mathbf{MLC} \cap \mathbf{MRI}$ *and* $\mathbf{MD}$ *are the three distinct maximal* $\mathcal{D}$*-compatible pseudovarieties of monoids and every* $\mathcal{D}$*-compatible pseudovariety of monoids is contained in one of these.*

As for the overcommutative case we remark that $\mathbf{MRI}_k = \mathbf{MLI}_k$ is determined by the identity $x^k y^k \approx y^k x^k$, and then

Theorem 7.6. *Theorem* 6.2 *remains valid for any overcommutative variety of monoids with* $\mathbf{RC}_k$ *replaced by* $\mathbf{MRC}_k$, P *by* P^1, P^* *by* $(P^*)^1$, Q *by* Q^1, Q^* *by* $(Q^*)^1$, $\mathbf{V}_{n,m}$ *by* $\mathbf{MV}_{n,m}$, $\mathbf{RC}_{n,m}$ *by* $\mathbf{MRC}_{n,m}$ *and* $\mathbf{LI}_{n,m}$ *by* $\mathbf{MLI}_{n,m}$.

Theorem 7.7. *If a* $\mathcal{D}$*-compatible monoid variety contains a nonabelian group, then it is a periodic variety which consists of completely regular monoids only.*

Theorem 3.15 allows to characterize and classify the monoid varieties and monoid pseudovarieties which are compatible for each of the Green relations.

8. $\mathcal{D} = \mathcal{H}$, $\mathcal{D} = \mathcal{L}$, $\mathcal{D} = \mathcal{R}$, $\mathcal{D} = \mathcal{J}$

If $\mathbf{V}$ is a variety such that $\mathcal{D} = \mathcal{H}$ for each $S \in \mathbf{V}$ then we shall say that $\mathbf{V}$ is *a variety for which* $\mathcal{D} = \mathcal{H}$, and we shall use a similar expression also in the cases where other Green relations are involved. In this section we investigate the semigroup varieties for which $\mathcal{D} = \mathcal{H}$ or $\mathcal{D} = \mathcal{L}$ or $\mathcal{D} = \mathcal{R}$ and we shall see that for such varieties we find that $\mathcal{D} = \mathcal{J}$. It also turns out that such varieties are aperiodic or $\mathcal{D}$-compatible.

We shall not give a systematic treatment of the semigroup varieties for which $\mathcal{D} = \mathcal{J}$. Of course every periodic semigroup variety is a variety for which $\mathcal{D} = \mathcal{J}$, and in Theorem 6.5 of [4] we have seen that $\mathcal{D} = \mathcal{J}$ for every cryptic semigroup variety. We shall see that there are $\mathcal{D}$-compatible (and thus by Theorem 6.2 of [5] and Theorem 6.2 also $\mathcal{L}$- or $\mathcal{R}$-compatible) semigroup varieties for which $\mathcal{D} = \mathcal{J}$ does not hold. This is all the more surprising since from Theorem 6.8 we know that a semigroup variety is $\mathcal{D}$-compatible if and only if it is $\mathcal{J}$-compatible. The $\mathcal{L}$-, $\mathcal{R}$- and $\mathcal{D}$-compatible semigroup varieties for which $\mathcal{D} = \mathcal{J}$ will be characterized and we shall find a semigroup variety which is minimal for not satisfying the property $\mathcal{D} = \mathcal{J}$.

Our first results will deal with the varieties for which $\mathcal{D} = \mathcal{R}$.

Recall that LZ_2 stands for a two element left zero semigroup. We define $\mathbf{NL}_{n,m}$ to be the subvariety of $\mathbf{V}_{n,m}$ determined by the identity

$$y^{mn}\left(xy^{mn}\right)^{mn} \approx \left(xy^{mn}\right)^{mn}. \tag{16}$$

The variety $\mathbf{NR}_{n,m}$ is defined in a dual way.

Lemma 8.1. $\mathbf{NL}_{n,m}$ *is the greatest subvariety of* $\mathbf{V}_{n,m}$ *which does not contain* LZ_2.

Proof. Clearly LZ_2 does not satisfy (16) and so $LZ_2 \notin \mathbf{NL}_{n,m}$. Let $\mathbf{V}$ be a subvariety of $\mathbf{V}_{n,m}$ which does not contain LZ_2 and let $S \in \mathbf{V}$. For $a, b \in S$, $e = b^{mn}$ and $f = (ab^{mn})^{mn}$ are idempotents of S such that $fe = f$. Therefore f and ef are $\mathcal{L}$-related idempotents of S. Since S cannot contain a copy of LZ_2 as a subsemigroup, we have that $f = ef$, and so S satisfies the identity (16). □

In accordance with the notation of [5], we let $\mathbf{LT}_n$ be the subvariety of $\mathbf{V}_{n,m}$ consisting of the semigroups of $\mathbf{V}_{n,m}$ whose regular $\mathcal{D}$-classes are right zero semigroups, or equivalently, $\mathbf{LT}_n$ consists of the semigroups of $\mathbf{V}_{n,m}$ for which the $\mathcal{L}$-relation is trivial. Thus, $\mathbf{LT}_n$ is the variety determined by the identities

$$y(xy)^n \approx (xy)^n, \qquad x^{n+1} \approx x^n. \tag{17}$$

The variety $\mathbf{RT}_n$ is defined dually.

Theorem 8.2.

(i) *For $n = 1$ or $m = 1$, $\mathbf{NL}_{n,m}$ is the greatest subvariety of $\mathbf{V}_{n,m}$ for which $\mathcal{D} = \mathcal{R}$.*
(ii) *For $m, m > 1$, the varieties $\mathbf{R}_{n,m} \cap \mathbf{NL}_{n,m}$, $\mathbf{RC}_{n,m} \cap \mathbf{NL}_{n,m}$ and $\mathbf{LT}_n$ are three distinct incomparable subvarieties of $\mathbf{V}_{n,m}$ for which $\mathcal{D} = \mathcal{R}$, and every subvariety of $\mathbf{V}_{n,m}$ for which $\mathcal{D} = \mathcal{R}$ is contained in one of these three.*

Proof. (i) $\mathbf{V}_{1,m}$ consists of semigroups which are completely regular. Since $\mathcal{D} = \mathcal{R}$ for a completely regular semigroup S if and only if S does not contains a copy of LZ_2 as a subsemigroup, it follows that $\mathbf{NL}_{1,m}$ is the greatest subvariety of $\mathbf{V}_{1,m}$ for which $\mathcal{D} = \mathcal{R}$. That $\mathbf{NL}_{n,1} = \mathbf{LT}_n$ is the greatest subvariety of $\mathbf{V}_{n,1}$ for which $\mathcal{D} = \mathcal{R}$ is obvious.

(ii) We now let $m, n > 1$ and investigate the subvarieties of $\mathbf{V}_{n,m}$ for which $\mathcal{D} = \mathcal{R}$. Clearly for $\mathbf{LT}_n$ we have $\mathcal{D} = \mathcal{R}$. Assume that $S \in \mathbf{R}_{n,m} \cap \mathbf{NL}_{n,m}$ and $a = tsa$ for some $a, t, s \in S$. By Theorem 4.1 of [5] $Gr(S)$ is a right ideal of S and since $a = (ts)^n a$ and $(ts)^n \in Gr(S)$, we have that $a \in Gr(S)$. The regular $\mathcal{D}$-classes of S are completely simple semigroups, and therefore $a\mathcal{L}sa$ in $Gr(S)$. Since $Gr(S)$ does not contain LZ_2 as a subsemigroup, it follows that $a\mathcal{H}sa$ in $Gr(S)$ and thus also in S. Therefore $\mathcal{H} = \mathcal{L}$ for S, hence $\mathcal{D} = \mathcal{R}$ for S. We now assume that $S \in \mathbf{RC}_{n,m} \cap \mathbf{NL}_{n,m}$ and $a = tsa$ for some $a, t, s \in S$. Since P does not divide S by Theorem 4.5 of [5], then S cannot have a divisor of the form A_2 or B_2, and so by Lemma 2.3, the regular $\mathcal{D}$-classes of S are completely simple. In fact, from Corollary 4.2 of [5] it follows that the regular $\mathcal{D}$-classes of S are abelian groups. Therefore $(ts)^{mn}t$ and $s(ts)^{mn}$ belong to the same maximal subgroup of S. Thus $a = (ts)^{mn}t(sa) \in s(ts)^{mn}S$ and $sa = s(ts)^{mn}S \in (ts)^{mn}tS$. Using Theorem 4.1 of [5] we thus have

$$sa = s(ts)^{mn}a = as(ts)^{mn},$$

$$a = (ts)^{mn}t(sa) = (sa)(ts)^{mn}t,$$

whence $sa\mathcal{R}a$. Therefore $\mathcal{H} = \mathcal{L}$ for S, thus also $\mathcal{D} = \mathcal{R}$ for S.

In the following we let $\mathbf{V}$ be a subvariety of $\mathbf{V}_{n,m}$ for which $\mathcal{D} = \mathcal{R}$. First assume that $Q \in \mathbf{V}$. Then $P \in \mathbf{V}$ since Q contains a subsemigroup isomorphic to P. Clearly $L_2(p) \notin \mathbf{V}$ for any prime p, and so from Lemma 2.13(ii) of [5] it follows that $\mathbf{V} \subseteq \mathbf{Ap}_n$. Also, since $A_2, B_2 \notin \mathbf{V}$ we have from Lemma 2.3 that for every $S \in \mathbf{V}$, the regular $\mathcal{D}$-classes of S are rectangular bands. Thus, $\mathbf{V} \subseteq \mathbf{LT}_n$ in this case.

Assume now that $\mathbf{V}$ is a subvariety of $\mathbf{V}_{n,m}$ for which $\mathcal{D} = \mathcal{R}$, and $Q \notin \mathbf{V}$. By Theorem 3.6 of [5], and since $\mathcal{D} = \mathcal{R}$ for $\mathbf{V}$, we have that $\mathbf{V}$ is $\mathcal{R}$-compatible. Thus by Theorem 4.6 of [5], $\mathbf{V}$ is contained in one of the three varieties $\mathbf{R}_{n,m} \cap \mathbf{NL}_{n,m}$, $\mathbf{RC}_{n,m} \cap \mathbf{NL}_{n,m}$ or $\mathbf{RAp}_n \cap \mathbf{NL}_{n,m}$. By Theorem 4.1 of [5], $\mathbf{RAp}_n \cap \mathbf{NL}_{n,m} \subseteq \mathbf{LT}_n$.

We still need to show that $\mathbf{R}_{n,m} \cap \mathbf{NL}_{n,m}$, $\mathbf{RC}_{n,m} \cap \mathbf{NL}_{n,m}$ and $\mathbf{LT}_n$ are three distinct incomparable subvarieties of $\mathbf{V}_{n,m}$. We see that Q belongs to $\mathbf{LT}_n$ but not to $\mathbf{R}_{n,m} \cap \mathbf{NL}_{n,m}$ nor to $\mathbf{RC}_{n,m} \cap \mathbf{NL}_{n,m}$. For any prime p, C_p belongs to $\mathbf{R}_{n,m} \cap \mathbf{NL}_{n,m}$ and $\mathbf{RC}_{n,m} \cap \mathbf{NL}_{n,m}$ but not to $\mathbf{LT}_n$. $\mathbf{RC}_{n,m} \cap \mathbf{NL}_{n,m}$ contains C, but $\mathbf{R}_{n,m} \cap \mathbf{NL}_{n,m}$ does not. $\mathbf{R}_{n,m} \cap \mathbf{NL}_{n,m}$ contains a nonabelian group, but $\mathbf{RC}_{n,m} \cap \mathbf{NL}_{n,m}$ does not by Corollary 4.2 of [5]. □

We now turn to the overcommutative semigroup varieties for which $\mathcal{D} = \mathcal{R}$. First we prove a few helpful lemmas concerning $\mathbf{RC}_k$ and $\mathbf{RC}_k \cap \mathbf{LI}_k$, $k > 0$.

Lemma 8.3. *For $k > 0$, $\mathbf{RC}_k$ is contained in the quasivariety determined by the implications*

$$xyz = z \quad \Rightarrow \quad z = zxy, \tag{18}$$

$$xyz = x \quad \Rightarrow \quad xuy = xyu, \tag{19}$$

$$xyz = y \quad \Rightarrow \quad y = yxz. \tag{20}$$

Proof. That $\mathbf{RC}_k$ satisfies (18) is the content of Lemma 6.1(ii), and that $\mathbf{RC}_k$ satisfies (19) follows from Theorem 6.2 of [5]. Now assume that $S \in \mathbf{RC}_k$, and $a = sat$ for some $a, s, t \in S$. Then $a = s^{2k}at^{2k} = s^k at^k (t^k s^k) = at^k s^k$ since $S \in \mathbf{RC}_k$. Since S satisfies (19) we have that $at^k p = apt^k$ for all $p \in S$, whence $a = at^k s^k = as^k t^k$. Again since S satisfies (19) we have that $asp = aps$ and $atp = apt$ for all $p \in S$. In particular then, $at^{k+1}s^{k+1} = at^k s^k (st)$. From the above and since $S \in \mathbf{RC}_k$ we thus have

$$\begin{aligned} a = s^{2k}at^{2k} &= s^{k-1}at^{k-1}t^{k+1}s^{k+1} \\ &= at^{k+1}s^{k+1} \\ &= at^k s^k st \\ &= ast. \end{aligned}$$

Thus $\mathbf{RC}_k$ satisfies (20). □

Lemma 8.4. *For $k > 0$, $\mathbf{RC}_k \cap \mathbf{LI}_k$ is contained in the quasivariety determined by the implications*

$$x_1x_2yz = y \quad \Rightarrow \quad ux_2y = uyx_2, \qquad x_1y = yx_1. \tag{21}$$

Proof. Let $S \in \mathbf{RC}_k \cap \mathbf{LI}_k$ and $tsap = a$ for some $a, s, t, p \in S$. Since S satisfies the implication (20) we have $a = atsp$, and since S satisfies the implication (19) we find that $atu = aut$, $asu = aus$ and $apu = aup$ for every $u \in S$. We have

$$\begin{aligned} a &= tsap \\ &= (ts)^{2k}tsap^{2k+1} \\ &= \cdots \\ &= (ts)^{2k}t^i s^i ap^{2k+i} \end{aligned}$$

$$
\begin{aligned}
&= (ts)^k t^i (ts)^k s^i ap^{2k+i} \\
&= t(st)^k t^i (st)^{k-1} s^{i+1} ap^{2k+i} \\
&= t(st)^{k+1} t^i (st)^{k-1} s^{i+1} ap^{2k+i+1} \\
&= t(st)^{2k} t^i s^{i+1} ap^{2k+i+1} \\
&= (ts)^{2k} t^{i+1} s^{i+1} ap^{2k+i+1} \\
&= \cdots \\
&= (ts)^{2k} t^k s^k ap^{3k} \\
&= ts(ts)^{2k} t^k s^k ap^{3k+1} \\
&= (ts)t^k s^k (ts)^{2k} ap^{3k+1} \quad (\text{since } S \in \mathbf{LI}_k) \\
&= (ts)t^k s^k ap^{k+1}.
\end{aligned}
$$

Put $(ts)t^k = w$, so that $ws^k a = a(st)^{k+1}$. Also

$$s^{k-1}as = s^{k-1}as^{k+1}p^k t^k = s^k as^k p^k t^k = s^k a$$

so that

$$s^{k-1+\ell}a = s^{k-1}as^{\ell} \quad \text{for all } \ell \geqslant 1.$$

Then

$$
\begin{aligned}
(sws^k)^k a &= s(ws^k)(sws^k)^{k-1} a \\
&= s(ws^{k+1})^{k-1} ws^k a \\
&= s(ws^{k+1})^{k-1} a(st)^{k+1} \\
&= s(ws^{k+1})^{k-2} ws^{k+1} a(st)^{k+1} \\
&= s(ws^{k+1})^{k-2} ws^k as(st)^{k+1} \\
&= s(ws^{k+1})^{k-2} as(st)^{2(k+1)} \\
&= \cdots \\
&= sas^{k-1}(st)^{k(k+1)}
\end{aligned}
$$

so that

$$
\begin{aligned}
(ws^k)^k (sws^k)^k a &= (ws^k)^k sas^{k-1}(st)^{k(k+1)} \\
&= (ws^k)^{k-1} ws^{k+1} as^{k-1}(st)^{k(k+1)} \\
&= (ws^k)^k as^k (st)^{k(k+1)} \\
&= as^k (st)^{2k(k+1)}.
\end{aligned}
$$

Therefore, and in view of the fact that $S \in \mathbf{LI}_k$, with $c \in S$

$$\begin{aligned} cas^k(st)^{2k(k+1)} &= c\big(ws^k\big)^k\big(sws^k\big)^k a \\ &= c\big(sws^k\big)^k\big(ws^k\big)^k a \\ &= c\big(sws^k\big)^k a(st)^{k(k+1)} \\ &= csas^{k-1}(st)^{2k(k+1)}. \end{aligned}$$

Multiplication on the right with $p^{k-1}t^{k-1}p^{2k(k+1)}$ then yields $cas = csa$.

In particular then $tsa = tas$, and $a = tsap = tasp$ so that $at = taspt = ta$. □

Theorem 8.5. *For an overcommutative semigroup variety* $\mathbf{V}$ *the following are equivalent*:

(i) $\mathcal{D} = \mathcal{R}$ *for every* $S \in \mathbf{V}$,
(ii) $\mathcal{J} = \mathcal{R}$ *for every* $S \in \mathbf{V}$,
(iii) $\mathbf{V}$ *is* $\mathcal{R}$*-compatible and* $LZ_2 \notin \mathbf{V}$,
(iv) $\mathbf{V}$ *is* $\mathcal{D}$*-compatible and* $P, LZ_2 \notin \mathbf{V}$,
(v) $\mathbf{V}$ *does not contain* LZ_2, $L_1(p)$, $L_2(p)$, *p prime*,
(vi) $\mathbf{V} \cap \mathbf{V}_{n,m} \subseteq \mathbf{RC}_{n,m} \cap \mathbf{NL}_{n,m}$ *for all* $n, m > 0$,
(vii) $\mathcal{D} = \mathcal{R}$ *for the periodic subvarieties of* $\mathbf{V}$,
(viii) *for some* $k > 0$, $\mathbf{V} \subseteq \mathbf{RC}_k$ *and* $\mathbf{V}$ *satisfies*

$$x^k y^k \approx y^k x^k, \tag{22}$$

(ix) $\mathbf{V}$ *is contained in the quasivariety determined by the implications*

$$xyz = z \quad \Rightarrow \quad yz = zy, \qquad xz = zx, \qquad z = zyx. \tag{23}$$

Proof. (ii) ⇒ (i) ⇒ (v) is obvious, and (iii) ⇔ (v) follows immediately from Theorem 6.3 of [4]. (v) ⇒ (vi) because of Lemma 8.1 and Theorem 6.2 of [5]. (vi) ⇒ (vii) follows from Theorem 8.2 and (vii) ⇒ (v) is obvious.

(iii) ⇒ (viii). Let $\mathbf{V}$ be $\mathcal{R}$-compatible and $LZ_2 \notin \mathbf{V}$. By Theorem 6.2 of [5], there exists $\ell > 0$ such that $\mathbf{V} \subseteq \mathbf{RC}_\ell$, and since $LZ_2 \notin \mathbf{V}$ there exists a balanced identity $xv \approx yw$ where $x, y \in X$ are distinct variables. We may assume that x and y are the only variables occurring in v and w, because otherwise we could obtain a consequence of this particular form using an appropriate substitution. We can also assume that $n_x(xv) = n_y(xv)$, because otherwise we could obtain a consequence of this particular form by multiplying on the right with an appropriate factor. Thus, $\mathbf{V}$ also satisfies the substitution instance $x^\ell v(x^\ell, y^\ell) \approx y^\ell w(x^\ell, y^\ell)$ of $xv \approx yw$. Let $k = \ell n_x(xv)$. Since $\mathbf{V}$ satisfies $x^{2\ell} y \approx x^\ell y x^\ell$ we find that $\mathbf{V}$ satisfies $x^k y^k \approx x^\ell v(x^\ell, y^\ell)$ and $y^k x^k \approx y^\ell w(x^\ell, y^\ell)$. Therefore $\mathbf{V}$ satisfies $x^k y^k \approx y^k x^k$. Also since $\mathbf{V} \subseteq \mathbf{RC}_\ell$ and $k \geqslant \ell$ we have that $\mathbf{V} \subseteq \mathbf{RC}_k$ by Proposition 6.1(ii) of [5].

(viii) ⇒ (iv) ⇒ (iii). Assume that $\mathbf{V} \subseteq \mathbf{RC}_k$ and $\mathbf{V}$ satisfies $x^k y^k \approx y^k x^k$ for some $k > 0$. Then $\mathbf{V}$ satisfies $zx^k y^k \approx zy^k x^k$ and so $\mathbf{V}$ is $\mathcal{D}$-compatible and $P \notin \mathbf{V}$ by Theorem 6.2. Also $LZ_2 \notin \mathbf{V}$ since LZ_2 does not satisfy $x^k y^k \approx y^k x^k$. That (iv) ⇒ (iii) follows immediately from Theorem 6.2 of [5] and Theorem 6.2.

(viii) $\Rightarrow$ (ii), (viii) $\Rightarrow$ (ix). We assume that $\mathbf{V} \subseteq \mathbf{RC}_k$ and that $\mathbf{V}$ satisfies $x^k y^k \approx y^k x^k$ for some $k > 0$. Let $S \in \mathbf{V}$ and assume that $a, b \in S$ such that $a\mathcal{J}b$ in S. Then $b = sar$ and $a = tbq$ for some $s, r, t, q \in S^1$. Putting $p = rq$ in S^1 we thus have that $a = tsap$. If $t = 1 = s$ or $t = 1 = p$, then obviously $a\mathcal{R}b$. If $t = 1$ and $s \neq 1 \neq p$, then $a = sap = s^2ap^2$ and so $sa = as$ since S satisfies (21). In this case we again have $a\mathcal{R}b$. By symmetry, if $s = 1$, then $a\mathcal{R}b$. We now assume that $s, t \in S$. By Lemma 8.3 we have that $a = tsap = apst$, with $pst \in S$. Therefore also by Lemma 8.3, $apu = aup$, $asu = aus$ and $atu = aut$ for every $u \in S$. From $a = tsap = tsap^2st$ we have $at = ta$ since S satisfies (21). Applying (21) and $at = ta$ repeatedly we find that $s^kt^ka = t^ks^ka = at^ks^k$. Therefore

$$s^k t^k a p^{k+1} st = a t^k s^k p^{k+1} st = a$$

and applying (21) again we see that $sa = as$. Therefore $bqt = sarqt = sapt = aspt = a$ and $asr = sar = b$ imply that $a\mathcal{R}b$. We proved that $\mathcal{R} = \mathcal{J}$ for S. To see that S satisfies the implications (23) it suffices to put $p = q = r = 1$ in the above computations.

(ix) $\Rightarrow$ (i). Let $\mathbf{V}$ be a variety which satisfies the implications (23), let $S \in \mathbf{V}$ and assume that $tsa = a$ for some $a, s, t \in S$. Then $sa = as$, $ta = at$ and $a = ast$. Consequently $sa\mathcal{R}a$. We proved that $\mathcal{H} = \mathcal{L}$ for S, and thus also $\mathcal{D} = \mathcal{R}$ for S. □

Corollary 8.6. *The semigroup varieties which are minimal for not satisfying the property $\mathcal{D} = \mathcal{R}$ for each of their members are the varieties each generated by one of the semigroups $L_1(p)$, $L_2(p)$, p a prime, or one of the semigroups B_2 or LZ_2. Every semigroup variety which does not satisfy $\mathcal{D} = \mathcal{R}$ for each of its members contains such a minimal one.*

Proof. From Lemma 2.12 of [5] we know that the varieties generated by the semigroups $L_1(p)$, $L_2(p)$, p prime, and the semigroup B_2 are pairwise incomparable. These varieties consist of semigroups with commuting idempotents and so they cannot contain LZ_2. Of course neither of these varieties can be contained in the atom $\mathbf{HSP}(LZ_2)$.

Let $\mathbf{V}$ be any semigroup variety which does not contain any of the varieties listed above. If $\mathbf{V}$ is overcommutative, then $\mathcal{D} = \mathcal{R}$ for $\mathbf{V}$ by Theorem 8.5. Now assume that $\mathbf{V} \subseteq \mathbf{V}_{n,m}$ for some $m, n > 0$. Since $B_2 \in \mathbf{V}$ then for every $S \in \mathbf{V}$ the regular $\mathcal{D}$-classes of S are completely simple by Lemma 2.3, and Lemma 2.11 of [5]. Since $LZ_2 \notin \mathbf{V}$ these regular $\mathcal{D}$-classes are in fact right groups and $\mathbf{V} \subseteq \mathbf{NL}_{n,m}$ by Lemma 8.1. If $\mathbf{V}$ is aperiodic then for every $S \in \mathbf{V}$ the regular $\mathcal{D}$-classes of S are right zero semigroups and $\mathbf{V} \subseteq \mathbf{LT}_n$. Otherwise $\mathbf{V}$ contains C_p for some prime p and so $P \notin \mathbf{V}$ by Lemma 2.13(ii) of [5]. A fortiori $Q \notin \mathbf{V}$ since Q contains a copy of P as a subsemigroup. By Theorem 3.6 of [5] we thus have that $\mathbf{V}$ is $\mathcal{R}$-compatible, and by Theorem 4.5 of [5] we have that $\mathbf{V} \subseteq \mathbf{R}_{n,m}$ or $\mathbf{V} \subseteq \mathbf{RC}_{n,m}$ or $\mathbf{V} \subseteq \mathbf{RAp}_n$. Since also $\mathbf{V} \subseteq \mathbf{NL}_{n,m}$ and $\mathbf{RAp}_n \cap \mathbf{NL}_{n,m} \subseteq \mathbf{LT}_n$ we thus have that $\mathbf{V} \subseteq \mathbf{R}_{n,m} \cap \mathbf{NL}_{n,m}$ or $\mathbf{V} \subseteq \mathbf{RC}_{n,m} \cap \mathbf{NL}_{n,m}$ or $\mathbf{V} \subseteq \mathbf{LT}_n$. By Theorem 8.2 we have that $\mathcal{D} = \mathcal{R}$ for $\mathbf{V}$. Since the varieties listed in the above statement are pairwise incomparable this also implies that every proper subvariety of any of these varieties satisfies the property that $\mathcal{D} = \mathcal{R}$ for its members. □

If $\mathbf{V}$ is a semigroup variety for which $\mathcal{D} = \mathcal{H}$, or equivalently, for which $\mathcal{L} = \mathcal{R}$, then $\mathbf{V}$ is compatible for each of the Green relations: this latter situation has been dealt with in Theorem 6.5 of [4] and Theorem 3.15. Clearly $\mathbf{V}$ is a semigroup variety for which $\mathcal{D} = \mathcal{H}$ if and only if $\mathbf{V}$ is the intersection of a variety for which $\mathcal{D} = \mathcal{R}$ and a variety for which $\mathcal{D} = \mathcal{L}$ and so the necessary information to characterize and classify such varieties may also be gathered from Theorems 8.2

and 8.5, Corollary 8.6 and their duals. We summarize the relevant results and omit the proofs. Especially Theorem 4.2 of [4] is useful to fill in the details.

Clearly $\mathbf{DT}_n = \mathbf{LT}_n \cap \mathbf{RT}_n$ is the subvariety of $\mathbf{V}_{n,m}$ consisting of all $S \in \mathbf{V}_{n,m}$ for which the $\mathcal{D}$-relation is trivial and $\mathbf{D}_{n,m} \cap \mathbf{NL}_{n,m} \cap \mathbf{NR}_{n,m}$ consists of the $S \in \mathbf{V}_{n,m}$ which are an ideal extension of a semilattice of groups by a nilsemigroup.

Theorem 8.7.

- (i) $\mathbf{D}_{1,m} \cap \mathbf{NL}_{1,m} \cap \mathbf{NR}_{1,m}$ *is the greatest subvariety of* $\mathbf{V}_{1,m}$ *for which* $\mathcal{D} = \mathcal{H}$ *and* $\mathbf{DT}_n$ *is the greatest subvariety of* $\mathbf{V}_{n,1}$ *for which* $\mathcal{D} = \mathcal{H}$. *For* $m, n > 1$, *the varieties* $\mathbf{D}_{n,m} \cap \mathbf{NL}_{n,m} \cap \mathbf{NR}_{n,m}$, $\mathbf{DT}_n$ *and* $\mathbf{C}_{n,m}$ *are the subvarieties of* $\mathbf{V}_{n,m}$ *which are maximal for satisfying the property* $\mathcal{D} = \mathcal{H}$ *for each of its members, and every subvariety of* $\mathbf{V}_{n,m}$ *satisfying the property that* $\mathcal{D} = \mathcal{H}$ *for each of its members is contained in one of these three.*
- (ii) *Let* $\mathbf{V}$ *be a subvariety of* $\mathbf{V}_{n,m}$ *satisfying the property* $\mathcal{D} = \mathcal{H}$ *for all its members. Then one of the following occur*:
  - (1) $\mathbf{V}$ *contains neither* C, P *nor* P^* *and then* $\mathbf{V} \subseteq \mathbf{D}_{n,m} \cap \mathbf{NL}_{n,m} \cap \mathbf{NR}_{n,m}$,
  - (2) $\mathbf{V}$ *contains* P *or* P^* *and then* $\mathbf{V} \subseteq \mathbf{DT}_n$,
  - (3) $\mathbf{V}$ *contains* C *but neither* P *nor* P^* *and then* $\mathbf{V} \subseteq \mathbf{C}_{n,m}$.
- (iii) *The semigroup varieties which are minimal for not satisfying the property* $\mathcal{D} = \mathcal{H}$ *for each of their members are the varieties each generated by one of the semigroups* $L_1(p), L_2(p), R_2(p)$, *p prime, or one of the semigroups* B_2, LZ_2 *or* RZ_2. *Every semigroup variety which does not satisfy* $\mathcal{D} = \mathcal{H}$ *for each of its members contains such a minimal one.*

We now turn our attention to semigroup varieties for which $\mathcal{D} = \mathcal{J}$. Since periodic semigroup varieties have this property we need to focus our efforts on the overcommutative situation. From Theorem 6.2 of [5] and Theorem 8.5 and its dual, $\mathcal{D} = \mathcal{J}$ is true for cryptic varieties and for varieties for which $\mathcal{D} = \mathcal{H}$ or $\mathcal{D} = \mathcal{L}$ or $\mathcal{D} = \mathcal{R}$. In the following we construct a semigroup which generates a $\mathcal{D}$-compatible variety for which $\mathcal{D} = \mathcal{J}$ does not hold.

Example. We let M consist of 0, and pairs and triples, bracketed or not, whose entries are integers. More precisely, apart from 0, M consists of the following pairs and triples:

$$\begin{aligned}
&(i, j, r), \quad && i, j, r \in \mathbb{Z}, \quad && i, j, r \geqslant 0,\ j + r > 0,\\
&[i, j, r], \quad && i, j, r \in \mathbb{Z}, \quad && i, j, r \geqslant 0,\\
&(i, j), \quad && i, j \in \mathbb{Z}, && \\
&[i, j], \quad && i, j \in \mathbb{Z}. &&
\end{aligned}$$

The multiplication of triples is defined by the following: for $i, j, r, l, m, t \geqslant 0$, put

$$\begin{aligned}
(i, j, r)(\ell, m, q) &= (i + \ell, j + m, r + q),\\
(i, j, r)[\ell, m, q] &= (i + \ell + 1, j + m, r + q),\\
[i, j, r](\ell, m, q) &= [i + \ell, j + m, r + q],\\
[i, j, r][\ell, m, q] &= [i + \ell + 1, j + m, r + q].
\end{aligned}$$

A multiplication of several factors each of which is a triple is clearly associative. The result of any other multiplication is either a pair or 0. We define for appropriate $i, j, \ell, m, t \in \mathbb{Z}$,

$$\begin{aligned}
(i, j)(\ell, m, q) &= (i + \ell - q, j + m - q),\\
(i, j)[\ell, m, q] &= (i + \ell + 1 - q, j + m - q),\\
[i, j](\ell, m, q) &= [i + \ell - q, j + m - q],\\
[i, j][\ell, m, q] &= [i + \ell + 1 - q, j + m - q],\\
(\ell, m, 0)(i, j) &= (\ell + i, m + j),\\
(\ell, m, 0)[i, j] &= (\ell + i + 1, m + j),\\
[\ell, m, 0](i, j) &= [\ell + i, m + j],\\
[\ell, m, 0][i, j] &= [\ell + i + 1, m + j],
\end{aligned}$$

and all other products of two factors are zero. A multiplication of several factors at least one of which is a pair is nonzero if and only if (i) only one of the factors is a pair, and (ii) any factor preceding this pair is a triple whose third entry is zero. If this is the case then the result of this multiplication is a pair, which is bracketed if and only if the first factor in the multiplication was bracketed. The entries in the end result are then $i + k - r$ and $j + k - r$, respectively, where i is the sum of the respective first entries in the factors, j the sum of the second entries, r the sum of the third entries and k the number of bracketed factors, disregarding the first factor. It follows that the multiplication defined above turns M into a semigroup.

If we put $s = [0, 0, 0]$, $t = (0, 1, 0)$, $p = (0, 0, 1)$ and $a = (0, 0)$, then

$$\begin{aligned}
(i, j, r) &= p^r s^i t^j,\\
[i, j, r] &= sp^r s^i t^j \quad \text{if } i \geqslant 1,\\
(i, j) &= as^i t^j \quad \text{if } i, j \geqslant 0,\\
&= ap^{-i-j} s^{-j} t^{-i} \quad \text{if } i, j \leqslant 0,\\
&= ap^{-j} s^{i-j} \quad \text{if } i \geqslant 0,\ j \leqslant 0,\\
&= ap^{-i} t^{j-i} \quad \text{if } i \leqslant 0,\ j \geqslant 0,\\
[i, j] &= s(i, j),\\
0 &= a^2,
\end{aligned}$$

where $p^0 = s^0 = t^0 = 1 \in M^1$. Therefore M is generated by $\{s, t, p, a\}$. Also

$$a(i, j, r)(\ell, m, q) = a(\ell, m, q)(i, j, r), \qquad a(i, j, r)[\ell, m, q] = a[\ell, m, q](i, j, r),$$
$$a[i, j, r][\ell, m, q] = a[\ell, m, q][i, j, r]$$

always hold, and also $a = apst$. Therefore the pairs (i, j), $i, j \in \mathbb{Z}$, form the $\mathcal{R}$-class of a, whereas the pairs $[i, j]$, $i, j \in \mathbb{Z}$, form the $\mathcal{R}$-class of sa, and all other $\mathcal{R}$-classes are trivial. For all $i, j \in \mathbb{Z}$ we have

$$M^1(i,j)=\{(\ell,m),[\ell,m],0 \mid \ell \geqslant i,\ m \geqslant j\},$$
$$M^1[i,j]=\{(\ell+1,m+1),[\ell,m],0 \mid \ell \geqslant i,\ m \geqslant j\},$$

from which we infer that the $\mathcal{L}$-relation on M is trivial, that is, $\mathcal{D}=\mathcal{R}$ for M. On the other hand, $a=tsap$, whence the pairs form the nontrivial $\mathcal{J}$-class of M. Clearly $\mathcal{J}\neq\mathcal{D}$ for M.

Lemma 8.8. *M satisfies the identity $v\approx w$ if and only if*

(i) *$v\approx w$ is a balanced identity,*
(ii) *$h(v)\approx h(w)$,*
(iii) *if $v=v_1yv_2$, $w=w_1yw_2$, $n_y(v)=1$, then $c(v_1)=c(w_1)$.*

Proof. We let $v\approx w$ be any identity which satisfies the conditions (i), (ii) and (iii) as stated, and let $\varphi: X\to M$ be any substitution. We want to show that $v\varphi=w\varphi$ holds in M. If $x\varphi\in\{p,s,t\}$ for every $x\in c(v)=c(w)$, then $v\varphi=w\varphi$ since the subsemigroup of M generated by p,s and t, that is, the subsemigroup of S consisting of the triples, is easily seen to satisfy the identity $xyz\approx xzy$. Otherwise $v\varphi=0=w\varphi$ unless the following occurs: $v=v_1yv_2$, $w=w_1yw_2$, $n_y(v)=n_y(w)=1$ and $y\varphi=a$, $x\varphi\in\{s,t\}$ for all $x\in c(v_1)=c(w_1)$ and $x\varphi\in\{p,s,t\}$ for all $x\in c(v_2)\cup c(w_2)$. If this is the case, then $v\varphi$ and $w\varphi$ will be pairs, and these pairs will be bracketed if and only if $h(v)\varphi=h(w)\varphi=s$. That $v\varphi=w\varphi$ then follows easily from the fact that in M

(i) $abc=acb$ for any b and c in the subsemigroup generated by p,s and t,
(ii) $ta=at$,
(iii) $bsa=bas$ for any $b\in M$.

We conclude that M satisfies every identity $v\approx w$ satisfying the conditions (i)–(iii) of the statement.

M satisfies balanced identities only since M contains an infinite cyclic subsemigroup, and if $v\approx w$ is satisfied by M then $h(v)=h(w)$ since LZ_2 is a homomorphic image of the subsemigroup generated by s and t. We now consider a balanced identity $v\approx w$ for which $h(v)=h(w)$ but which fails to satisfy the condition (iii). We may assume that $v=v_1yv_2$, $w=w_1yw_2$, $n_y(v)=n_y(w)=1$ and $x\in c(v_1)-c(w_1)$. We use a substitution $\varphi: X\to M$ for which $x\varphi=p$, $y\varphi=a$ and $z\varphi=t$ otherwise. Then $v\varphi=0\neq w\varphi$. We proved that M satisfies only identities which satisfy the stated conditions. □

Corollary 8.9. **HSP**(M) *is a subvariety of the $\mathcal{D}$-compatible variety* $\mathbf{RC}_2\cap\mathbf{LI}_2$ *and* **HSP**(M) *is not a variety for which $\mathcal{D}=\mathcal{J}$.*

Lemma 8.10. *Let* $\mathbf{V}$ *be a subvariety of* $\mathbf{RC}_\ell$ *for some $\ell>1$ and let* $\mathbf{V}$ *satisfy a balanced identity $v\approx w$ for which $h(v)=h(w)$ but which fails to satisfy condition* (iii) *of Lemma* 8.8. *Then* $\mathbf{V}$ *satisfies an identity of the form*

$$x^kz^ky\approx x^kyz^k \tag{24}$$

for some $k>0$.

Proof. We may assume that $\mathbf{V}$ satisfies an identity $v \approx w$ where $v = v_1 y v_2$, $w = w_1 y w_2$, $h(v) = h(w) = x$, $n_y(v) = 1$ and $z \in c(v_1) - c(w_1)$. We substitute z by z^ℓ, y by y and all other variables by x^ℓ, apply the identity $x^{2\ell} y \approx x^\ell y x^\ell$ repeatedly and obtain a consequence of the form

$$x^{m\ell} z^{n\ell} y \approx x^{m\ell} y z^{n\ell}, \tag{25}$$

which is satisfies by $\mathbf{V}$. We put $k = \ell mn$ and obtain (24) as a consequence from (25) by substituting x by x^n, y by y and z by z^m. $\square$

We are now in the position to characterize the $\mathcal{D}$-compatible semigroup varieties for which $\mathcal{D} = \mathcal{J}$.

Theorem 8.11. *Let* $\mathbf{V}$ *be an overcommutative semigroup variety. Then* $\mathbf{V}$ *is an* $\mathcal{R}$*-compatible variety for which* $\mathcal{D} = \mathcal{J}$ *if and only if one of the following or their duals occur*:

(i) $\mathbf{V}$ *satisfies the equivalent conditions of Theorem* 8.5*, or*
(ii) *for some* $k > 0$*,* $\mathbf{V}$ *is contained in the subvariety of* $\mathbf{RC}_k$ *determined by the identity* (25).

Proof. Assume that $\mathbf{V}$ is an overcommutative $\mathcal{R}$-compatible variety for which $\mathcal{D} = \mathcal{J}$. By Theorem 6.2 we have that $\mathbf{V} \subseteq \mathbf{RC}_\ell$ for some $\ell > 0$. By Corollary 8.9, $\mathbf{V}$ does not contain M, and since $\mathbf{V}$ satisfies balanced identities only, it follows that $\mathbf{V}$ satisfies a balanced identity $v \approx w$ which fails to satisfy condition (ii) or condition (iii) of Lemma 8.8. If $v \approx w$ is such that $h(v) \neq h(w)$, then $\mathbf{V}$ is $\mathcal{R}$-compatible by Theorem 6.2 of [4] and since $\mathbf{V} \subseteq \mathbf{RC}_\ell$, and $\mathbf{V}$ does not contain LZ_2, hence $\mathbf{V}$ satisfies the equivalent conditions of Theorem 8.5. If on the other hand, $v \approx w$ is balanced and $h(v) = h(w)$ but condition (iii) of Lemma 8.8 fails to hold true, then $\mathbf{V}$ satisfies an identity $x^m z^m y \approx x^m y z^m$ for some $m > 0$ by Lemma 8.10. Putting $k = \ell m$, we then have that (ii) holds.

If $\mathbf{V}$ satisfies (i) then by Theorem 8.5, $\mathbf{V}$ is an $\mathcal{R}$-compatible variety for which $\mathcal{D} = \mathcal{R} = \mathcal{J}$. Now assume that for some $k > 0$, $\mathbf{V} \subseteq \mathbf{RC}_k$ and $\mathbf{V}$ satisfies $x^k z^k y \approx x^k y z^k$. Let $S \in \mathbf{V}$ and let $a, b \in S$ such that $a \mathcal{J} b$. Then $b = sar$ and $a = tbq$ for some $s, r, t, q \in S^1$, thus $a = tsarq$. If $rq = 1 \in S^1$ then $a \mathcal{L} sa \mathcal{R} sar = b$ and if $ts = 1 \in S^1$ then $a \mathcal{R} ar \mathcal{L} sar = b$. Otherwise we put $m = ts$, $p = rq$ and we have that $a = map$ with $m, p \in S$. Since $S \in \mathbf{RC}_k$ and satisfies the identity (24) we find

$$\begin{aligned} a = map &= m^{2k} a p^{2k} \\ &= m^{2k} p^k a p^k \\ &= m^{k-1} p^k m^{k+1} a p^k \\ &= m^{k-1} p^k m^k (ma) p^k \\ &= m^{k-1} p^k m^k p^k (ma) \end{aligned}$$

and so $a \mathcal{L} ma$ and a fortiori $a \mathcal{L} sa$. From Lemma 8.3 we further have $a = apm$ from $a = map$. Thus $a \mathcal{R} ar$ and so also $sa \mathcal{R} sar = b$. We conclude that $\mathcal{D} = \mathcal{J}$ for S. $\square$

Theorem 8.12.

(i) *The variety generated by M is the smallest $\mathcal{R}$-compatible variety which does not satisfy $\mathcal{D} = \mathcal{J}$ for each of its members.*
(ii) *The $\mathcal{D}$-compatible varieties minimal for not satisfying $\mathcal{D} = \mathcal{J}$ for each of their members are the varieties each generated by M or its dual M^*. Every $\mathcal{D}$-compatible variety which does not satisfy $\mathcal{D} = \mathcal{J}$ for each of its members contains such a minimal one.*

Proof. The assertion follows immediately from Lemma 8.8, Corollary 8.9, Lemma 8.10 and Theorem 8.11. □

Remark. The results of Theorems 8.2 and 8.5 and Corollary 8.6 find their analogues for pseudovarieties of semigroups and for varieties and pseudovarieties of monoids and we shall not spell out the details. The subsequent results however are irrelevant for overcommutative varieties of monoids, as the following theorem indicates.

Theorem 8.13. *Let $\mathbf{V}$ be a $\mathcal{D}$-compatible overcommutative monoid variety. Then $\mathbf{V}$ is $\mathcal{R}$-compatible and $\mathcal{J} = \mathcal{R}$ or $\mathbf{V}$ is $\mathcal{L}$-compatible and $\mathcal{J} = \mathcal{L}$.*

Proof. By Theorems 6.2 and 7.6 and using duality we may assume that there exists $k > 0$ such that $\mathbf{V} \subseteq \mathbf{MRC}_k \cap \mathbf{MLI}_k$. Since $\mathbf{LI}_k$ is determined by the identity (11), $\mathbf{MLI}_k$ is determined by the identity (22). Thus $\mathbf{V}$ is a monoid variety for which $\mathcal{J} = \mathcal{R}$ by Theorem 8.5 and since its semigroup reducts belong to the subvariety of $\mathbf{RC}_k$ determined by the identity (22). □

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